Minimal locally solid topologies on Riesz spaces

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(X, t) is a Hausdorff topological vector space.

(X, t) (or t) is said to be minimal if there does not exists a Hausdorff topology on X strictly coarser than t.

Minimal spaces are complete.

Minimality is preserved by closed subspaces and finite products.

If Y is a minimal subspace of X and the quotient X/Y is minimal, than X has to be minimal.

The algebraic sum of two closed subspaces $Y, Z \subset X$ may be non-closed. But if $Y \cap Z = \{0\}$ and Y or Z is minimal than $Y + Z$ remains closed.

The algebraic sum of two transversal minimal subspaces is a minimal space.

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The algebraic sum of two transversal minimal subspaces is a minimal space.

(Schaefers book): A locally convex space (X, t) is minimal if and only if it is isomorphic, i.e., linearly homeomorphic, to $\mathbb{K}^\Gamma$ for the field $\mathbb{K}$ of real or complex numbers and some nonempty set Γ.

An example of a minimal non locally convex space ?

Quotient-minimal space (X, t) = all quotients of X by closed subspaces are minimal.

Riesz space (=vector lattice) E

A linear topology τ on E is locally solid (and (E, τ) is said to be a locally solid Riesz space) if τ has a base of neighborhoods of zero consisting of solid sets U, i.e., $y \in U$ and $|x| \leqslant |y|$ implies $x \in U$.

Locally solid τ is generated by a family of monotone F-seminorms $(p_\alpha)_{\alpha\in\mathcal{A}}$: $|x| \leqslant |y| \;\Rightarrow\; p_\alpha(x) \leqslant p_\alpha(y)$.

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Some Riesz spaces E do not admit monotone F-norms.

Consider $E = \mathbb{R}^\Gamma$ with uncountable Γ; suppose p is a monotone F-norm on E. There exist (γ_n) and a natural k such that $p(e_{\gamma_n}) > \frac{1}{k}$ where e_γ = γ's unit vector. Define $f \in E$ as: $f(\gamma_n) = n$ and $f(\gamma) = 0$ for $\gamma \in \Gamma \setminus \{\gamma_n : n \in \mathbb{N}\}$. Clearly $f = \sup_n ne_{\gamma_n}$. Therefore $p(e_{\gamma_n}) \leqslant p(\frac{1}{n}f) \to 0$, a contradiction.

There exist Riesz spaces without any Hausdorff locally solid topology. Let E be the universal completion of $C[0,1]$. If τ were a Hausdorff locally solid topology on E, then it would be a σ-Lebesgue, i.e., $x_n \downarrow 0 \;\Rightarrow\; x_n \xrightarrow{\tau} 0$. Since $C[0,1]$ is order dense in its universal completion, the restriction $\tau|_{C[0,1]}$ is also σ-Lebesgue and Hausdorff. But $C[0,1]$ (W.A.J. Luxemburg) does not admit any Hausdorff σ-Lebesgue topology.

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A Hausdorff locally solid Riesz space (E, τ) is minimal if E does not admit a Hausdorff locally solid topology essentially coarser than τ.

The minimality is preserved by projection bands in E.

An example of an F-lattice (= complete metrizable locally solid Riesz space) whose topology is the coarsest Hausdorff locally solid topology on it.

(C. Aliprantis and O. Burkinshaw) - if (S, Σ, μ) is a σ-finite measure space, then the topology of convergence in measure on sets of finite measure is the coarsest Hausdorff locally solid topology on $L^0(\mu)$.

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Proof

(L. Drewnowski) - assumption: $\mu(S) < \infty$.

$\tau(\mu)$ = the topology of measure convergence; τ = a Hausdorff topology on $L^0(\mu)$. Since $\tau(\mu)$ is metrizable and complete, then $\tau \subseteq \tau(\mu)$. The topology τ is generated by a family p_α of monotone F-seminorms, and their kernels $N_\alpha = \{f \in L^0(\mu) : p_\alpha(f)\}$ are $\tau(\mu)$-closed ideals because $\tau \subseteq \tau(\mu)$. Thus they are bands, and so $N_\alpha = \{f \in L^0(\mu) : \operatorname{supp} f \subset S_\alpha\}$ where S_α are some elements from Σ. It means exactly that $N_\alpha = \{1_{S_\alpha}\}^{dd}$. A family of characteristic functions of sets from Σ forms a complete Boolean algebra with the property that every subset having the supremum contains a countable subset with the same supremum.

Therefore, for some measurable set A and some sequence of indices (α_n) there holds $1_A = \sup_\alpha 1_{S \setminus S_\alpha} = \sup_n 1_{S \setminus S_{\alpha_n}}$. Thus $S \setminus A \subset S_\alpha$ (μ-almost everywhere) and since $\bigcap_\alpha N_\alpha = \{0\}$, then $\mu(S \setminus A) = 0$. Hence $1_S = 1_A$ μ-almost everywhere. We get $1_S = \sup_n 1_{S \setminus S_{\alpha_n}} = 1_S - \inf_n 1_{S_{\alpha_n}}$. Now if $p_{\alpha_n}(f) = 0$ for all n's, then $\operatorname{supp} f \subset \bigcap_n S_{\alpha_n}$, and so $1_{\operatorname{supp} f} \leqslant \inf_n 1_{S_{\alpha_n}} = 0$ what means $\mu(\operatorname{supp} f) = 0$, i.e., $f = 0$. We have just shown the sequence od F-seminorms (p_{α_n}) is total. Hence there exists a monotone F-norm p generating the same topology as (p_{α_n}). Apply an Drewnowski's result: a monotone F-norm on $L^0(\mu)$ defines a topology finer than $\tau(\mu)$. We obtain: $\tau(\mu) \subset \tau(p) = \tau((p_{\alpha_n})) \subset \tau((p_\alpha)) = \tau \subset \tau(\mu)$. Finally $\tau = \tau(\mu)$.

We showed more: the topology of convergence in measure is a unique Hausdorff locally solid topology admitted on $L^0(\mu)$.

N.T. Peck: for an atomless measure μ the space $L^0(\mu)$ admits a Hausdorff linear topology coarser than the topology of convergence in measure.

Aliprantis and Burkinshaw, monograph "Locally solid Riesz spaces with Applications to Economics":

Theorem

If a Riesz space E admits a complete metrizable Lebesgue topology, then E has the weakest locally solid topology.

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Therefore $L^p(\mu)$-spaces, $p \in (0, \infty)$, possess the weakest locally solid Hausdorff topology - $\tau(\mu)$ is the weakest locally solid Hausdorff topology on L^p. The same is true for Orlicz spaces.

$\tau(\mu)$ is not weakest on $L^\infty(\mu)$ but this topology is minimal on $L^\infty(\mu)$.

A Riesz space E is Archimedean whenever $\forall_{0 \leqslant x \in E} \ \inf_n \frac{1}{n} x = 0$.

Theorem

An Archimedean Riesz space E admits the weakest locally solid topology which is locally convex iff E is discrete - in which case the weakest topology is the topology of pointwise convergence.

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The discreteness of E means: E can be embedded, with preservation of the order structure, into $\mathbb{R}^\Gamma$ in such a way that the image of E contains unit vectors: span$\{1_{\{\gamma\}} : \gamma \in \Gamma\} \subset E \subset \mathbb{R}^\Gamma$.

$(L^0(\mu), \tau(\mu)$ has very strong order properties:

$\tau(\mu)$ is unique Hausdorff locally solid topology admitted by $L^0(\mu)$,

$L^0(\mu)$ coincides with its universal completion and so the space is laterally complete and super Dedekind complete (i.e., families of pairwise disjoint elements have a supremum and every set bounded from above has a supremum and contains a countable subset having the same supremum), The discreteness of E means: E can be embedded, with preservation of the order structure, into $\mathbb{R}^{\Gamma}$ in such a way that the image of E contains unit vectors: $\text{span}\{1_{\{\gamma\}} : \gamma \in \Gamma\} \subset E \subset \mathbb{R}^{\Gamma}$.

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$L^0(\mu)$ has a weak unit (i.e., an element $e \geqslant 0$ such that $|x| \wedge e = 0 \;\Rightarrow\; x = 0$ - clearly in our case we have a lot of weak units: every measurable function f positive almost everywhere is a weak unit $f(s) > 0$ for μ-almost everywhere),

$\tau(\mu)$ is a Lebesgue topology ($x_\alpha \downarrow 0 \;\Rightarrow\; x_\alpha \xrightarrow{\tau(\mu)} 0$),

$\tau(\mu)$ is a Levi topology, i.e, topologically bounded increasing nets of positive elements have a supremum.

Levi topologies are always complete.

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A locally solid topology τ determines another locally solid topology $\tau_{\mathfrak{u}}$: take a base $(U_i)_{i\in I}$ of solid τ-neighborhoods of zero and for an arbitrary positive $u \in E$ put

$$U_{i,u} = \{x \in E : |x| \wedge u \in U_i\}.$$

The family $(U_{i,u})_{i\in I, u\geqslant 0}$ forms a base of neighborhoods at zero for a (formally weaker than τ) locally solid topology $\tau_{\mathfrak{u}}$ on E and $\tau_{\mathfrak{u}}$ is Hausdorff whenever τ is Hausdorff.

M.A. Taylor:

Theorem

For a Hausdorff locally solid topology τ on a Riesz space E the following statements are equivalent

(i) *τ is Lebesgue and $\tau = \tau_{\mathfrak{u}}$,*

(ii) *τ is minimal.*

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Quotient Riesz spaces E/F have their own peculiarity; one must always assume that F is an ideal.

(E, τ) is quotient-by-ideals-minimal whenever E/F is minimal for every closed ideal F in E.

An F-lattice (E, τ) is quotient-by-ideals-minimal if and only if τ is minimal.

Indeed, if τ is minimal, the topology τ is Lebesgue, and so closed ideals F in E are projection bands. Therefore E/F is order isomorphic to the projection band F^{d} which is a minimal locally solid Riesz space.

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$\dim E > 1 \;\Rightarrow\;$ E contains Riesz subspaces F, G with $F + G$ is not a Riesz subspace; if F or G is an ideal, then $F + G$ is a Riesz subspace.

Is $F + G$ closed whenever $F, G \subset (E, \tau)$, F is a closed ideal, G is a closed Riesz subspace ?

Theorem

Let I be a closed infinite dimensional and infinite codimensional ideal in an F-lattice E. If I does not contain any order copy of $\mathbb{R}^{\mathbb{N}}$ then there exists a closed separable Riesz subspace G such that $I \cap G = \{0\}$ and $I + G$ is not closed.

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On the other hand

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Let I be a closed ideal in an F-lattice (E, τ). If $F \subset E$ is a closed Riesz subspace and τ restricted to F is minimal then $I + F$ is closed.

Question: is the sum $I + F$ closed whenever F is a closed Riesz subspace and I is an ideal, both included in an F-lattice (E, τ), and τ restricted to I is minimal ?

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Proof of the Proposition

Suppose first that $I \cap F = \{0\}$. The quotient map $Q : E \to E/I$ restricted to F is a continuous injective Riesz homomorphism, and so it is a homeomorphism. Hence $Q(F)$ is closed in E/I because $Q(F)$ is complete. The continuity of Q implies $I + F = Q^{-1}(Q(F))$ is closed.

Let $I \cap F = F_1 \neq \{0\}$. The subspace F_1 is a closed ideal in a Dedekind complete Riesz space F and τ restricted to F is a Lebesgue topology. F_1 is a projection band in F. If F_1^{d} denotes the disjoint complement of F_1 in F then F_1^{d} is minimal and $I \cap F_1^{\mathrm{d}} = \{0\}$. The first part of the proof shows $I + F_1^{\mathrm{d}} = I + F$ is closed.