Extension operators for smooth functions on compact subsets of the reals

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Joint work with Leonhard Frerick and Enrique Jordá

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Universität Trier

The Smooth Extension Property (SEP)

For a compact set $K \subseteq \mathbb{R}^d$ the restriction space is

$$C^\infty(K) = \{F|_K : F \in C^\infty(\mathbb{R}^d)\}$$

endowed with the quotient topology of $C^\infty(\mathbb{R}^d)$ given by the norms

$$\|f\|_n = \inf\{\sup\{|\partial^\alpha F(x)| : x \in \mathbb{R}^d, |\alpha| \leq n\} : F|_K = f\}.$$

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- Semi-coherent subanalytic sets have SEP (Bierstone-Milman 1998) ($K = \varphi(M)$ where φ is real-analytic on a real-analytic manifold and $\mathscr{E}(\mathbb{R}^d) \to \mathscr{E}(M)$, $F \mapsto F \circ \varphi$ has closed range)

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- Quotient norms are hard to calculate

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- $E(f)(x, y) = f(x, 0) + y \exp(1/x) \left(f(x, \exp(-1/x)) - f(x, 0)\right)$

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Idea: partition of unity $\rightsquigarrow$ local problems on small intervals J, if the ideal prescribes many zeros for f on J approximate by 0, otherwise define g by Hermite interpolation with appropriate data depending on f and $n \leq m \leq k$. In fact, $m = 2n + 1$ and $s = k$ always work.

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- For $X_n = \{f \in C^n(\mathbb{R}) : f|_K = 0\}$ the proof shows $X_m \subseteq \overline{X_k}^{C^n(\mathbb{R})}$ for all $k \geq m$

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Every closed ideal in $C^\infty(\mathbb{R})$ has (Ω).

- $K \subseteq \mathbb{R}$ has SEP $\Leftrightarrow C^\infty(K)$ has (DN)
- For $X_n = \{f \in C^n(\mathbb{R}) : f|_K = 0\}$ the proof shows $X_m \subseteq \overline{X_k}^{C^n(\mathbb{R})}$ for all $k \geq m$
- For $K \subseteq \mathbb{R}$ always $C^\infty(K) = \bigcap_{n \in \mathbb{N}} C^n(K)$ (Merrien 1966)

When does $C^\infty(K)$ have (DN)?

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A Fréchet space with semi-norms $\|\cdot\|_n$ has (DN) if

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A case study: $K = \{x_n : n \in \mathbb{N}\} \cup \{0\}$

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On the other hand, $\{0\} \cup \bigcup_{n\in\mathbb{N}} \left[\exp(-p^n), \exp(-p^n) + \exp(-p^{n+1})\right]$ does have WEP and SEP! (Goncharov 1996 for $p \in \{3, 4, 5, ...\}$)

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