Contributions to Analysis and Functional Analysis

in memoriam

Paweł Domański

Dietmar Vogt

Bergische Universität Wuppertal

Paweł Domański Memorial Conference
Bedlewo 2018

Contributions to:

- Real analytic functions
  - Linear structure, subspaces, quotient spaces, bases
  - Algebra structure: ideals, algebra/composition operators
  - Vector valued real analytic functions
- Homological theory, Ext, Proj
  - Interplay with vector valued real analytic functions, solutions with parameter, interpolation
  - Special invariants, exactness of tensorized sequences
  - Splitting of differential complexes
- Special operators
  - Structural theory of Hadamard-/Euler-operators on real analytic functions
  - Solvability of Hadamard-/Euler-operators on real analytic or smooth functions

Contributions to:

- Real analytic functions
  - Linear structure, subspaces, quotient spaces, bases
  - Algebra structure: ideals, algebra/composition operators
  - Vector valued real analytic functions
- Homological theory, Ext, Proj
  - Interplay with vector valued real analytic functions, solutions with parameter, interpolation
  - Special invariants, exactness of tensorized sequences
  - Splitting of differential complexes
- Special operators
  - Structural theory of Hadamard-/Euler-operators on real analytic functions
  - Solvability of Hadamard-/Euler-operators on real analytic or smooth functions

Many more results on:

Theory of functional analysis, dynamics of operators, vector-valued hyper-functions, abstract Cauchy-problem, ...

Real analytic functions

Let $\Omega \subset \mathbb{R}^d$ be open. A real analytic function is an infinitely differentiable function which can be expanded around every point in Ω into its Taylor series or, equivalently, which can be extended to a holomorphic function on some complex neighborhood of Ω.

Real analytic functions

Let $\Omega \subset \mathbb{R}^d$ be open. A real analytic function is an infinitely differentiable function which can be expanded around every point in Ω into its Taylor series or, equivalently, which can be extended to a holomorphic function on some complex neighborhood of Ω.

$A(\Omega)$ denotes the space of real analytic functions on Ω. It is an algebra over $\mathbb{C}$. It carries a unique natural topology which makes it a topological algebra. This is given by

$$A(\Omega) = \mathrm{proj}_n H(K_n) = \mathrm{ind}_\omega H(\omega)$$

where $K_1 \subset K_2 \subset \dots$ is a compact exhaustion of Ω and ω runs through the complex neighborhoods of Ω.

Real analytic functions

Let $\Omega \subset \mathbb{R}^d$ be open. A real analytic function is an infinitely differentiable function which can be expanded around every point in Ω into its Taylor series or, equivalently, which can be extended to a holomorphic function on some complex neighborhood of Ω.

$A(\Omega)$ denotes the space of real analytic functions on Ω. It is an algebra over $\mathbb{C}$. It carries a unique natural topology which makes it a topological algebra. This is given by

$$A(\Omega) = \mathrm{proj}_n H(K_n) = \mathrm{ind}_\omega H(\omega)$$

where $K_1 \subset K_2 \subset \ldots$ is a compact exhaustion of Ω and ω runs through the complex neighborhoods of Ω.

With this topology $A(\Omega)$ is complete, nuclear, ultrabornological (PDF)-space.

Fréchet sub- and quotient spaces of $A(\Omega)$

E Fréchet space, $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \dots$ semi-norms.

Fréchet sub- and quotient spaces of $A(\Omega)$

E Fréchet space, $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \ldots$ semi-norms.

- E subspace of $A(\Omega) \Rightarrow E \in (\underline{DN})$.
- E quotient space of $A(\Omega) \Rightarrow E \in (\overline{\overline{\Omega}})$.

Fréchet sub- and quotient spaces of $A(\Omega)$

E Fréchet space, $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \ldots$ semi-norms.

- E subspace of $A(\Omega) \Rightarrow E \in (\underline{DN})$.
- E quotient space of $A(\Omega) \Rightarrow E \in (\overline{\overline{\Omega}})$.

Examples?

Fréchet sub- and quotient spaces of $A(\Omega)$

E Fréchet space, $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \ldots$ semi-norms.

- E subspace of $A(\Omega) \Rightarrow E \in (\underline{DN})$.
- E quotient space of $A(\Omega) \Rightarrow E \in (\overline{\overline{\Omega}})$.

EXAMPLES?
Fréchet subspaces: zero solutions of an elliptic operator

Fréchet sub- and quotient spaces of $A(\Omega)$

E Fréchet space, $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \ldots$ semi-norms.

- E subspace of $A(\Omega) \Rightarrow E \in (\underline{DN})$.
- E quotient space of $A(\Omega) \Rightarrow E \in (\overline{\overline{\Omega}})$.

EXAMPLES?
Fréchet subspaces: zero solutions of an elliptic operator
Fréchet quotient spaces: they exist (difficult to show)

Fréchet sub- and quotient spaces of $A(\Omega)$

E Fréchet space, $\|\cdot\|_1 \le \|\cdot\|_2 \le \dots$ semi-norms.

- E subspace of $A(\Omega) \Rightarrow E \in (\underline{DN})$.
- E quotient space of $A(\Omega) \Rightarrow E \in (\overline{\overline{\Omega}})$.

EXAMPLES?
Fréchet subspaces: zero solutions of an elliptic operator
Fréchet quotient spaces: they exist (difficult to show)

- Precise characterization of Fréchet subspaces and quotients by invariants and nuclearity-type

Fréchet sub- and quotient spaces of $A(\Omega)$

E Fréchet space, $\|\cdot\|_1 \le \|\cdot\|_2 \le \dots$ semi-norms.

- E subspace of $A(\Omega) \Rightarrow E \in (\underline{DN})$.
- E quotient space of $A(\Omega) \Rightarrow E \in (\overline{\overline{\Omega}})$.

Examples?
Fréchet subspaces: zero solutions of an elliptic operator
Fréchet quotient spaces: they exist (difficult to show)

- Precise characterization of Fréchet subspaces and quotients by invariants and nuclearity-type

Theorem: E complemented Fréchet subspace of $A(\Omega) \Rightarrow E$ finite dimensional.

Fréchet sub- and quotient spaces of $A(\Omega)$

E Fréchet space, $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \ldots$ semi-norms.

- E subspace of $A(\Omega) \Rightarrow E \in (\underline{DN})$.
- E quotient space of $A(\Omega) \Rightarrow E \in (\overline{\overline{\Omega}})$.

Examples?
Fréchet subspaces: zero solutions of an elliptic operator
Fréchet quotient spaces: they exist (difficult to show)

- Precise characterization of Fréchet subspaces and quotients by invariants and nuclearity-type

Theorem: E complemented Fréchet subspace of $A(\Omega) \Rightarrow E$ finite dimensional.

Consequence: Improvement of the Grothendieck-Poly result.

Bases in $A(\Omega)$

A basis is a sequence $f_1, f_2, f_3, \ldots$ in $A(\Omega)$ so that every $f \in A(\Omega)$ has a unique expansion $f = \sum_{n=1}^{\infty} \lambda_n f_n$.

Bases in $A(\Omega)$

A basis is a sequence $f_1, f_2, f_3, \ldots$ in $A(\Omega)$ so that every $f \in A(\Omega)$ has a unique expansion $f = \sum_{n=1}^{\infty} \lambda_n f_n$.

$A(\Omega)$ is a complete separable locally convex space.
Long term problem: does it have a basis, and if so, what is it?

Bases in $A(\Omega)$

A basis is a sequence $f_1, f_2, f_3, \ldots$ in $A(\Omega)$ so that every $f \in A(\Omega)$ has a unique expansion $f = \sum_{n=1}^{\infty} \lambda_n f_n$.

$A(\Omega)$ is a complete separable locally convex space.
Long term problem: does it have a basis, and if so, what is it?

Theorem: $A(\Omega)$ has no basis.

Bases in $A(\Omega)$

A basis is a sequence $f_1, f_2, f_3, \ldots$ in $A(\Omega)$ so that every $f \in A(\Omega)$ has a unique expansion $f = \sum_{n=1}^{\infty} \lambda_n f_n$.

$A(\Omega)$ is a complete separable locally convex space.
Long term problem: does it have a basis, and if so, what is it?

Theorem: $A(\Omega)$ has no basis.

ARGUMENT:

- If $A(\Omega)$ has a basis then it is a (PLS)-Köthe space.
- An ultrabornological (PLS)-Köthe without infinite dimensional complemented Fréchet subspaces is a (DF)-space.
- $A(\Omega)$ has no infinite dimensional complemented Fréchet subspaces but it is not a (DF)-space.
- $A(\Omega)$ has no basis.

Ideals in the algebra $A(\mathbb{R}^d)$

$J \subset A(\mathbb{R}^d)$ closed ideal.

Ideals in the algebra $A(\mathbb{R}^d)$

$J \subset A(\mathbb{R}^d)$ closed ideal.
For an analytic set V in neighborhood of $\mathbb{R}^d$ define:
$J_V = \{f \in A(\mathbb{R}^d) \ : f_a = 0 \text{ on } V_a \text{ for all } a \in X = V \cap \mathbb{R}^d\}.$

Ideals in the algebra $A(\mathbb{R}^d)$

$J \subset A(\mathbb{R}^d)$ closed ideal.
For an analytic set V in neighborhood of $\mathbb{R}^d$ define:
$J_V = \{f \in A(\mathbb{R}^d) \ : \ f_a = 0$ on V_a for all $a \in X = V \cap \mathbb{R}^d\}$.

Results:

- There is V such that $J = J_V$ iff $J_a = Rad(J_a)$ for all $a \in X$.

Ideals in the algebra $A(\mathbb{R}^d)$

$J \subset A(\mathbb{R}^d)$ closed ideal.
For an analytic set V in neighborhood of $\mathbb{R}^d$ define:
$J_V = \{f \in A(\mathbb{R}^d) \ : \ f_a = 0$ on V_a for all $a \in X = V \cap \mathbb{R}^d\}$.

Results:

- There is V such that $J = J_V$ iff $J_a = Rad(J_a)$ for all $a \in X$.
- If J_V is complemented in $A(\mathbb{R}^d)$ then V_a satisfies PL_{loc} for all $a \in X$.
- If X is compact or homogeneous then the converse is true.

Ideals in the algebra $A(\mathbb{R}^d)$

$J \subset A(\mathbb{R}^d)$ closed ideal.
For an analytic set V in neighborhood of $\mathbb{R}^d$ define:
$J_V = \{f \in A(\mathbb{R}^d) \ : \ f_a = 0 \text{ on } V_a \text{ for all } a \in X = V \cap \mathbb{R}^d\}$.

Results:

- There is V such that $J = J_V$ iff $J_a = Rad(J_a)$ for all $a \in X$.
- If J_V is complemented in $A(\mathbb{R}^d)$ then V_a satisfies PL_{loc} for all $a \in X$.
- If X is compact or homogeneous then the converse is true.

V_a satisfies PL_{loc} if there is a constant $A > 0$ such that for all holomorphic functions f on V_a

$$|f(z)| \leq \|f\|_{V_a}^{A|\mathrm{Im}\, z|} \, \|f\|_{X_a}^{1-A|\mathrm{Im}\, z|}$$

in a complex neighborhood of a independent of f.

Composition operators

M, N real analytic manifolds, algebra hom. $A(N) \to A(M)$:

Composition operators

M, N real analytic manifolds, algebra hom. $A(N) \to A(M)$:
$\varphi : M \to N$ real analytic map, $C_\varphi(f) := f \circ \varphi$.

Composition operators

M, N real analytic manifolds, algebra hom. $A(N) \to A(M)$:
$\varphi : M \to N$ real analytic map, $C_\varphi(f) := f \circ \varphi$.
$J := \ker C_\varphi$. $X = \mathrm{Loc} J$

- C_φ has closed range $\Rightarrow$ $\varphi(M)$ has the global extension property (and $\varphi(M) = X$).
- Converse is true if φ semi-proper.

Composition operators

M, N real analytic manifolds, algebra hom. $A(N) \to A(M)$:
$\varphi : M \to N$ real analytic map, $C_\varphi(f) := f \circ \varphi$.
$J := \ker C_\varphi$. $X = \mathrm{Loc} J$

- C_φ has closed range $\Rightarrow$ $\varphi(M)$ has the global extension property (and $\varphi(M) = X$).
- Converse is true if φ semi-proper.
- C_φ is open onto its range $\Leftrightarrow$ φ is semi-proper and $\varphi(M)$ has the semi-global extension property.

Composition operators

M, N real analytic manifolds, algebra hom. $A(N) \to A(M)$:
$\varphi : M \to N$ real analytic map, $C_\varphi(f) := f \circ \varphi$.
$J := \ker C_\varphi$. $X = \mathrm{Loc} J$

- C_φ has closed range $\Rightarrow$ $\varphi(M)$ has the global extension property (and $\varphi(M) = X$).
- Converse is true if φ semi-proper.
- C_φ is open onto its range $\Leftrightarrow$ φ is semi-proper and $\varphi(M)$ has the semi-global extension property.

Special case $C_\varphi : A(\mathbb{R}) \to A(\mathbb{R})$.

- Complete description of the eigenvalues and the eigenvectors of C_φ.

Composition operators

M, N real analytic manifolds, algebra hom. $A(N) \to A(M)$:
$\varphi : M \to N$ real analytic map, $C_\varphi(f) := f \circ \varphi$.
$J := \ker C_\varphi$. $X = \mathrm{Loc} J$

- C_φ has closed range $\Rightarrow$ $\varphi(M)$ has the global extension property (and $\varphi(M) = X$).
- Converse is true if φ semi-proper.
- C_φ is open onto its range $\Leftrightarrow$ φ is semi-proper and $\varphi(M)$ has the semi-global extension property.

Special case $C_\varphi : A(\mathbb{R}) \to A(\mathbb{R})$.

- Complete description of the eigenvalues and the eigenvectors of C_φ.
- The Abel equation $C_\varphi f = f + 1$ has a real analytic solution iff φ has no fixed points and the set of critical points is bounded from above/below.

Homological concepts (local/global problem)

$$X = \lim\ \mathrm{proj}_j X_j,\ Y = \lim\ \mathrm{proj}_j Y_j,\ Z = \lim\ \mathrm{proj}_j Z_j,$$

Homological concepts (local/global problem)

$X = \lim\ \mathrm{proj}_j X_j$, $Y = \lim\ \mathrm{proj}_j Y_j$, $Z = \lim\ \mathrm{proj}_j Z_j$,
exact sequence

$$0 \to X \to Y \xrightarrow{q} Z$$

Homological concepts (local/global problem)

$X = \lim \text{proj}_j X_j$, $Y = \lim \text{proj}_j Y_j$, $Z = \lim \text{proj}_j Z_j$, exact sequence

$$0 \to X \to Y \xrightarrow{q} Z$$

Assume: induces for all j exact sequences

$$0 \to X_j \to Y_j \xrightarrow{q_j} Z_j \to 0$$

Homological concepts (local/global problem)

$X = \lim\ \mathrm{proj}_j X_j$, $Y = \lim\ \mathrm{proj}_j Y_j$, $Z = \lim\ \mathrm{proj}_j Z_j$, exact sequence

$$0 \to X \to Y \xrightarrow{q} Z$$

Assume: induces for all j exact sequences

$$0 \to X_j \to Y_j \xrightarrow{q_j} Z_j \to 0$$

Problem: q surjective?

Homological concepts (local/global problem)

$X = \lim\ \mathrm{proj}_j X_j,\ Y = \lim\ \mathrm{proj}_j Y_j,\ Z = \lim\ \mathrm{proj}_j Z_j,$
exact sequence

$$0 \to X \to Y \xrightarrow{q} Z$$

Assume: induces for all j exact sequences

$$0 \to X_j \to Y_j \xrightarrow{q_j} Z_j \to 0$$

Problem: q surjective?
Yields "long exact sequence"

$$0 \to X \to Y \xrightarrow{q} Z \to \mathrm{Proj}^1 X \to \mathrm{Proj}^1 Y \to \mathrm{Proj}^1 Z \to 0.$$

Homological concepts (local/global problem)

$X = \lim\ \mathrm{proj}_j X_j$, $Y = \lim\ \mathrm{proj}_j Y_j$, $Z = \lim\ \mathrm{proj}_j Z_j$,
exact sequence

$$0 \to X \to Y \xrightarrow{q} Z$$

Assume: induces for all j exact sequences

$$0 \to X_j \to Y_j \xrightarrow{q_j} Z_j \to 0$$

Problem: q surjective?
Yields "long exact sequence"

$$0 \to X \to Y \xrightarrow{q} Z \to \mathrm{Proj}^1 X \to \mathrm{Proj}^1 Y \to \mathrm{Proj}^1 Z \to 0.$$

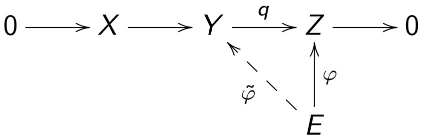
EXAMPLE: $\Omega \subset \mathbb{R}^d$ open, $P(\partial)$ differential operator, $K_1 \subset K_2 \subset$ compact exhaustion, $P(\partial) : \mathscr{E}(K_j) \to \mathscr{E}(K_j)$ surjective for all j, $C^\infty(\Omega) = \lim\ \mathrm{proj}_j \mathscr{E}(K_j)$, $P(\partial) : C^\infty(\Omega) \to C^\infty(\Omega)$ surjective?

Lifting, splitting, tensors

All of the following are based on Proj^1-concept.

Lifting, splitting, tensors

All of the following are based on Proj^1-concept.



Lifting, splitting, tensors

All of the following are based on Proj^1-concept.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & & & & \nwarrow{\scriptstyle \tilde{\varphi}} & \uparrow{\scriptstyle \varphi} & & \\ & & & & & & E & & \end{array}$$

Assume that for all j there is $\tilde{\varphi}_j$ with

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_j & \longrightarrow & Y_j & \xrightarrow{q_j} & Z_j & \longrightarrow & 0 \\ & & & & & \nwarrow{\scriptstyle \tilde{\varphi}_j} & \uparrow{\scriptstyle \varphi_j} & & \\ & & & & & & E & & \end{array}$$

Lifting, splitting, tensors

All of the following are based on Proj^1-concept.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \xrightarrow{q} & Z & \longrightarrow & 0 \\
 & & & & & {\scriptstyle\tilde{\varphi}}\nwarrow & \uparrow{\scriptstyle\varphi} & & \\
 & & & & & & E & &
\end{array}$$

Assume that for all j there is $\tilde{\varphi}_j$ with

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_j & \longrightarrow & Y_j & \xrightarrow{q_j} & Z_j & \longrightarrow & 0 \\
 & & & & & {\scriptstyle\tilde{\varphi}_j}\nwarrow & \uparrow{\scriptstyle\varphi_j} & & \\
 & & & & & & E & &
\end{array}$$

Yields for every j exact sequence

$$0 \longrightarrow L(E, X_j) \longrightarrow L(E, Y_j) \longrightarrow L(E, Z_j) \longrightarrow 0$$

and therefore a long exact sequence

$$0 \to L(E,X) \to L(E,Y) \to L(E,Z) \to \mathrm{Ext}^1(E,X) \to \mathrm{Ext}^1(E,Y) \to ..$$

and therefore a long exact sequence

$$0 \to L(E,X) \to L(E,Y) \to L(E,Z) \to \mathrm{Ext}^1(E,X) \to \mathrm{Ext}^1(E,Y) \to ..$$

$\mathrm{Ext}^1(E,X) = 0$ implies under suitable assumptions:

1. Every $\varphi \in L(E,Z)$ can be lifted under q to $\tilde{\varphi} \in L(E,Y)$
2. $q \otimes \mathrm{id} : Y \hat{\otimes}_\pi E' \to Z \hat{\otimes}_\pi E'$ is surjective
3. Every exact sequence $0 \to X \to Y \to E \to 0$ splits.

and therefore a long exact sequence

$$0 \to L(E,X) \to L(E,Y) \to L(E,Z) \to \mathrm{Ext}^1(E,X) \to \mathrm{Ext}^1(E,Y) \to ..$$

$\mathrm{Ext}^1(E,X) = 0$ implies under suitable assumptions:

1. Every $\varphi \in L(E,Z)$ can be lifted under q to $\tilde{\varphi} \in L(E,Y)$
2. $q \otimes \mathrm{id} : Y \hat{\otimes}_\pi E' \to Z \hat{\otimes}_\pi E'$ is surjective
3. Every exact sequence $0 \to X \to Y \to E \to 0$ splits.

Conditions for $\mathrm{Ext}^1(E,X) = 0$, X (PLS)-space:

- For E Fréchet - or (DF) space, in terms of splitting conditions (H), resp. (G)
- Same, in terms of invariants $(P*)$ for X and suitable $*$ for E, where $*$ stands for one of the submultiplicative invariants of Fréchet space theory.

and therefore a long exact sequence

$$0 \to L(E,X) \to L(E,Y) \to L(E,Z) \to \mathrm{Ext}^1(E,X) \to \mathrm{Ext}^1(E,Y) \to ..$$

$\mathrm{Ext}^1(E,X) = 0$ implies under suitable assumptions:

1. Every $\varphi \in L(E,Z)$ can be lifted under q to $\tilde{\varphi} \in L(E,Y)$
2. $q \otimes \mathrm{id} : Y \hat{\otimes}_\pi E' \to Z \hat{\otimes}_\pi E'$ is surjective
3. Every exact sequence $0 \to X \to Y \to E \to 0$ splits.

Conditions for $\mathrm{Ext}^1(E,X) = 0$, X (PLS)-space:

- For E Fréchet - or (DF) space, in terms of splitting conditions (H), resp. (G)
- Same, in terms of invariants $(P*)$ for X and suitable $*$ for E, where $*$ stands for one of the submultiplicative invariants of Fréchet space theory.

Condition for $\mathrm{Proj}^1(X \hat{\otimes}_\pi E) = 0$, X and E (PLN)-spaces, in terms of condition (T).

Splitting condition, Invariants

$X = \lim \operatorname{proj}_N X_N$; $X_N = \lim \operatorname{ind} X_{N,n}$, with norms $\| \cdot \|_{N,n}$,
E analogous.

Splitting condition, Invariants

$X = \lim \operatorname{proj}_N X_N$; $X_N = \lim \operatorname{ind} X_{N,n}$, with norms $\|\cdot\|_{N,n}$,
E analogous.

Condition (T):

$$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m\ \exists\ k, S\ \forall\ x \in X_n', y \in E_N':$$
$$\|x \circ i_n^M\|_{M,m}^* \|y \circ i_n^M\|_{M,m}^* \leq S\left(\|x\|_{N,n}^*\|y\|_{N,n}^* + \|x \circ i_N^K\|_{K,k}^*\|y \circ i_N^K\|_{K,k}^*\right).$$

Dual interpolation property for small θ:

$$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m\ \exists\ \theta_0 \in]0,1[\ \forall\ \theta \in]0,\theta_0[\ \exists\ k, C,\ \forall\ x' \in X_N':$$
$$\|x' \circ i_n^M\|_{M,m}^* \leq C\,\|x' \circ i_N^K\|_{K,k}^{*(1-\theta)}\|x'\|_{N,n}^{*\theta}.$$

Property $(P\overline{\overline{\Omega}})$:

$$\forall\ N\ \exists\ M \geq N\ \forall\ K \geq M\ \exists\ n\ \forall\ m\ \exists\ k, C, \theta \in]0,1[\ \forall\ x' \in X_N':$$
$$\|x' \circ i_n^M\|_{M,m}^* \leq C\,\|x' \circ i_N^K\|_{K,k}^{*(1-\theta)} \max\left(\|x'\|_{N,n}^{*\theta}, \|x' \circ i_N^K\|_{K,k}^{*\theta}\right).$$

Vector valued real analytic functions

Let E be a complete locally convex space. An E-valued function f on $\mathbb{R}^d$ is called real analytic if $y \circ f$ is real analytic for every $y \in E'$.

Notation: $A(\mathbb{R}^d, E)$.

Vector valued real analytic functions

Let E be a complete locally convex space. An E-valued function f on $\mathbb{R}^d$ is called real analytic if $y \circ f$ is real analytic for every $y \in E'$.

Notation: $A(\mathbb{R}^d, E)$.

Then $A(\mathbb{R}^d, E) \; = E \hat{\otimes}_\pi A(\mathbb{R}^d)$.

Vector valued real analytic functions

Let E be a complete locally convex space. An E-valued function f on $\mathbb{R}^d$ is called real analytic if $y \circ f$ is real analytic for every $y \in E'$.

Notation: $A(\mathbb{R}^d, E)$.

Then $A(\mathbb{R}^d, E) \;= E\hat{\otimes}_\pi A(\mathbb{R}^d)$.

Problem: Can every $f \in A(\mathbb{R}^d, E)$ be expanded in Taylor series around all x?

Answer: If $E \in$ (DF): Yes; if E Fréchet : $\Leftrightarrow$ $E \in$(DN).

Vector valued real analytic functions

Let E be a complete locally convex space. An E-valued function f on $\mathbb{R}^d$ is called real analytic if $y \circ f$ is real analytic for every $y \in E'$.

Notation: $A(\mathbb{R}^d, E)$.

Then $A(\mathbb{R}^d, E) \; = E\hat{\otimes}_\pi A(\mathbb{R}^d)$.

Problem: Can every $f \in A(\mathbb{R}^d, E)$ be expanded in Taylor series around all x?

Answer: If $E \in$ (DF): Yes; if E Fréchet : $\Leftrightarrow$ $E \in$(DN).

Problem of interpolation:

E fixed, surjective restriction map $\rho : A(\mathbb{R}^d) \; \to A(X)$. Is $\rho : A(\mathbb{R}^d, E) \; \to A(X, E)$ surjective?
E. g. $X \in \mathbb{R}^d$ discrete, $\ker \rho \cong A(\mathbb{R}^d)$.

Vector valued real analytic functions

Let E be a complete locally convex space. An E-valued function f on $\mathbb{R}^d$ is called real analytic if $y \circ f$ is real analytic for every $y \in E'$.

Notation: $A(\mathbb{R}^d, E)$.

Then $A(\mathbb{R}^d, E) \; = E \hat{\otimes}_\pi A(\mathbb{R}^d)$.

Problem: Can every $f \in A(\mathbb{R}^d, E)$ be expanded in Taylor series around all x?

Answer: If $E \in$ (DF): Yes; if E Fréchet : $\Leftrightarrow$ $E \in$(DN).

Problem of interpolation:

E fixed, surjective restriction map $\rho : A(\mathbb{R}^d) \; \to A(X)$. Is
$\rho : A(\mathbb{R}^d, E) \; \to A(X, E)$ surjective?
E. g. $X \in \mathbb{R}^d$ discrete, $\ker \rho \cong A(\mathbb{R}^d)$.
Answer: If E Fréchet space, then surjective; If E (DF)-space, then surjective iff $E \in (\underline{A})$.

Solvability with parameters

$P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ surjective. U real analytic manifold.

$f \in A(U, \mathscr{D}'(\Omega))$, i.e.for $\lambda \in U$ $f_\lambda \in \mathscr{D}'(\Omega)$, λ "parameter"

Problem: Does there exist for every f a function $g \in A(U, \mathscr{D}'(\Omega))$ such that $P(D)g_\lambda = f_\lambda$ for all λ.

Solvability with parameters

$P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ surjective. U real analytic manifold.

$f \in A(U, \mathscr{D}'(\Omega))$, i.e.for $\lambda \in U$ $f_\lambda \in \mathscr{D}'(\Omega)$, λ "parameter"

Problem: Does there exist for every f a function $g \in A(U, \mathscr{D}'(\Omega))$ such that $P(D)g_\lambda = f_\lambda$ for all λ.

Theorem: Yes iff $\ker P(D)$ has dual interpolation estimate for small θ (U non-compact connected) or has property $(P\overline{\overline{\Omega}})$ (U compact).

Solvability with parameters

$P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ surjective. U real analytic manifold.

$f \in A(U, \mathscr{D}'(\Omega))$, i.e.for $\lambda \in U$ $f_\lambda \in \mathscr{D}'(\Omega)$, λ "parameter"

Problem: Does there exist for every f a function $g \in A(U, \mathscr{D}'(\Omega))$ such that $P(D)g_\lambda = f_\lambda$ for all λ.

Theorem: Yes iff $\ker P(D)$ has dual interpolation estimate for small θ (U non-compact connected) or has property $(P\overline{\overline{\Omega}})$ (U compact).

$\rightsquigarrow$ Translation of invariants into properties of polynomial P via PL-conditions.

Solvability with parameters

$P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ surjective. U real analytic manifold.

$f \in A(U, \mathscr{D}'(\Omega))$, i.e.for $\lambda \in U$ $f_\lambda \in \mathscr{D}'(\Omega)$, λ "parameter"

Problem: Does there exist for every f a function $g \in A(U, \mathscr{D}'(\Omega))$ such that $P(D)g_\lambda = f_\lambda$ for all λ.

Theorem: Yes iff $\ker P(D)$ has dual interpolation estimate for small θ (U non-compact connected) or has property $(P\overline{\overline{\Omega}})$ (U compact).

$\rightsquigarrow$ Translation of invariants into properties of polynomial P via PL-conditions.

Theorem: P homogeneous , Ω open convex. Solvability with real analytic parameters iff $P(D)$ has a continuous linear right inverse in $\mathscr{D}'(\Omega)$ (equivalently: in $C^\infty(\Omega)$).

Splitting of differential complexes

Fréchet spaces can be graded, i. e. written as projective limit in different ways: $\| \cdot \|_n$ fundamental system of semi-norms in E.

1. E_n 'Banach grading',
2. $E/\| \cdot \|_n$ with quotient topology 'strict norm grading'.

Splitting of differential complexes

Fréchet spaces can be graded, i. e. written as projective limit in different ways: $\|\cdot\|_n$ fundamental system of semi-norms in E.

1. E_n 'Banach grading',
2. $E/\|\cdot\|_n$ with quotient topology 'strict norm grading'.

$\Omega \subset \mathbb{R}^d$ open, $K_1 \subset K_2 \subset ..$ compact exhaustion, $\mathscr{E}(K_n)$ is strict norm grading, $\mathrm{Proj}^1/\mathrm{Ext}^1$-method yields:

Splitting of differential complexes

Fréchet spaces can be graded, i. e. written as projective limit in different ways: $\|\cdot\|_n$ fundamental system of semi-norms in E.
1. E_n 'Banach grading',
2. $E/\|\cdot\|_n$ with quotient topology 'strict norm grading'.

$\Omega \subset \mathbb{R}^d$ open, $K_1 \subset K_2 \subset ..$ compact exhaustion, $\mathscr{E}(K_n)$ is strict norm grading, $\mathrm{Proj}^1/\mathrm{Ext}^1$-method yields:

Theorem: If $\Omega_i \subset \mathbb{R}^d$ are open subsets and $T_0 : C^\infty(\Omega_0)^s \to C^\infty(\Omega_1)^{s_1}$ is a matrix of convolution operators such that the corresponding complex

$$0 \to \ker T_0 \to C^\infty(\Omega_0)^s \stackrel{T_0}{\to} C^\infty(\Omega_1)^{s_1} \stackrel{T_1}{\to} C^\infty(\Omega_2)^{s_2} \to \dots,$$

is algebraically exact, then the complex splits at $C^\infty(\Omega_k)^{s_k}$ for $k = 1, 2, \dots$. It splits at $C^\infty(\Omega_0)^s$ iff $\ker T_0$ is strict graded.

Splitting of differential complexes

Fréchet spaces can be graded, i. e. written as projective limit in different ways: $\|\cdot\|_n$ fundamental system of semi-norms in E.

1. E_n 'Banach grading',
2. $E/\|\cdot\|_n$ with quotient topology 'strict norm grading'.

$\Omega \subset \mathbb{R}^d$ open, $K_1 \subset K_2 \subset ..$ compact exhaustion, $\mathscr{E}(K_n)$ is strict norm grading, $\mathrm{Proj}^1/\mathrm{Ext}^1$-method yields:

Theorem: If $\Omega_i \subset \mathbb{R}^d$ are open subsets and $T_0 : C^\infty(\Omega_0)^s \to C^\infty(\Omega_1)^{s_1}$ is a matrix of convolution operators such that the corresponding complex

$$0 \to \ker T_0 \to C^\infty(\Omega_0)^s \stackrel{T_0}{\to} C^\infty(\Omega_1)^{s_1} \stackrel{T_1}{\to} C^\infty(\Omega_2)^{s_2} \to \ldots,$$

is algebraically exact, then the complex splits at $C^\infty(\Omega_k)^{s_k}$ for $k = 1, 2, \ldots$. It splits at $C^\infty(\Omega_0)^s$ iff $\ker T_0$ is strict graded.

Analogous result where $C^\infty(\Omega)^s$ is replaced with $\mathscr{D}'(\Omega)^s$.

Hadamard operators

$\Omega \subset \mathbb{R}^d$ open, $E(\Omega) = A(\Omega)$ or $C^\infty(\Omega)$.

Definition: A map $L \in L(E(\Omega))$ is called a Hadamard operator ($L \in M(E(\Omega))$) if it admits all monomials as eigenvectors.

Hadamard operators

$\Omega \subset \mathbb{R}^d$ open, $E(\Omega) = A(\Omega)$ or $C^\infty(\Omega)$.

Definition: A map $L \in L(E(\Omega))$ is called a Hadamard operator ($L \in M(E(\Omega))$) if it admits all monomials as eigenvectors.

- $L \in M(E(\Omega))$: $L(x^\alpha) = m_\alpha x^\alpha$.
- $L \mapsto (m_\alpha)_{\alpha \in \mathbb{N}^d}$ injective algebra homomorphism.
- $\Rightarrow M(E(\Omega))$ is a closed commutative subalgebra of $L(E(\Omega))$.

Hadamard operators

$\Omega \subset \mathbb{R}^d$ open, $E(\Omega) = A(\Omega)$ or $C^\infty(\Omega)$.

Definition: A map $L \in L(E(\Omega))$ is called a Hadamard operator ($L \in M(E(\Omega))$) if it admits all monomials as eigenvectors.

- $L \in M(E(\Omega))$: $L(x^\alpha) = m_\alpha x^\alpha$.
- $L \mapsto (m_\alpha)_{\alpha \in \mathbb{N}^d}$ injective algebra homomorphism.
- $\Rightarrow M(E(\Omega))$ is a closed commutative subalgebra of $L(E(\Omega))$.

Representation theorem: Hadamard operators on $E(\Omega)$ have the form $(M_T f)(y) = T_x f(xy)$, where $T \in E'$, supp $T \subset V(\Omega)$.

Notations: Multiplication $xy = (x_1 y_1, \ldots, x_d y_d)$
$V(\Omega) = \{x \, : \, xy \in \Omega \text{ for all } y \in \Omega\}$ 'dilation set'

Hadamard operators

$\Omega \subset \mathbb{R}^d$ open, $E(\Omega) = A(\Omega)$ or $C^\infty(\Omega)$.

Definition: A map $L \in L(E(\Omega))$ is called a Hadamard operator ($L \in M(E(\Omega))$) if it admits all monomials as eigenvectors.

- $L \in M(E(\Omega))$: $L(x^\alpha) = m_\alpha x^\alpha$.
- $L \mapsto (m_\alpha)_{\alpha \in \mathbb{N}^d}$ injective algebra homomorphism.
- $\Rightarrow M(E(\Omega))$ is a closed commutative subalgebra of $L(E(\Omega))$.

Representation theorem: Hadamard operators on $E(\Omega)$ have the form $(M_T f)(y) = T_x f(xy)$, where $T \in E'$, supp $T \subset V(\Omega)$.

NOTATIONS: Multiplication $xy = (x_1 y_1, \ldots, x_d y_d)$
$V(\Omega) = \{x : xy \in \Omega \text{ for all } y \in \Omega\}$ 'dilation set'

Consequences: 1. For $L = M_T$: $m_\alpha = T x^\alpha$
2.$M_T \circ M_S = M_{T \star S}$ where $(T \star S) f = T_x(S_y f(xy))$
3. the Cauchy transform $C_T \in \mathscr{O}_0$, $C_{T\star S} = C_T \odot C_S$ where $\odot$ denotes the Hadamard product.

Euler operators

Objects of study for Hadamard operators:

- $M(E(\Omega))$ as topological algebra.
- Dilation sets.
- Global solvability of $M_T u = g$.

Euler operators

Objects of study for Hadamard operators:

- $M(E(\Omega))$ as topological algebra.
- Dilation sets.
- Global solvability of $M_T u = g$.

Results on solvability for special case:

Euler partial differential operators:

$$P(\theta) = \sum_{\alpha} c_\alpha \theta^\alpha, \qquad \theta_j = x_j \partial / \partial x_j$$

Euler operators

Objects of study for Hadamard operators:

- $M(E(\Omega))$ as topological algebra.
- Dilation sets.
- Global solvability of $M_T u = g$.

Results on solvability for special case:

Euler partial differential operators:

$$P(\theta) = \sum_{\alpha} c_\alpha \theta^\alpha, \qquad \theta_j = x_j \partial / \partial x_j$$

On $Q =]0, +\infty[^d$: $P(\theta) x^z = P(z) x^z$ for all $z \in \mathbb{C}^d$, $\Rightarrow$ $m_\alpha = P(\alpha)$.

Mellin transform $\langle f(x), x^z \rangle$ replaces Fourier transform.

Euler operators

Objects of study for Hadamard operators:

- $M(E(\Omega))$ as topological algebra.
- Dilation sets.
- Global solvability of $M_T u = g$.

Results on solvability for special case:

Euler partial differential operators:

$$P(\theta) = \sum_\alpha c_\alpha \theta^\alpha, \qquad \theta_j = x_j \partial / \partial x_j$$

On $Q =]0, +\infty[^d$: $P(\theta)x^z = P(z)x^z$ for all $z \in \mathbb{C}^d$, $\Rightarrow$ $m_\alpha = P(\alpha)$.

Mellin transform $\langle f(x), x^z \rangle$ replaces Fourier transform.

PROBLEM: defined only on Q. Needs extended theory.

Global solutions

P polynomial, P_m principal part.

Global solutions

P polynomial, P_m principal part.

Case of $E(\Omega) = A(\Omega)$, $\Omega = \mathbb{R}^d$

Necessary condition: $P(\theta)u = g \Rightarrow$ (*) $\partial^\alpha g(0) = 0$ if $P(\alpha) = 0$.

Global solutions

P polynomial, P_m principal part.

Case of $E(\Omega) = A(\Omega)$, $\Omega = \mathbb{R}^d$

Necessary condition: $P(\theta)u = g \Rightarrow$ (*) $\partial^\alpha g(0) = 0$ if $P(\alpha) = 0$.

Assume: $P_m(e_j) \neq 0$ all j, e_j unit vectors, P has finitely many integer zeros.

Theorem: $P(\theta)u = g$ has a solution for all $g \in A(\mathbb{R}^d)$ with (*) $\Leftrightarrow P_m(z) \neq 0$ for all $z \in \mathbb{C}^d$ with $\operatorname{Re} z \in Q$.

Global solutions

P polynomial, P_m principal part.

Case of $E(\Omega) = A(\Omega)$, $\Omega = \mathbb{R}^d$

Necessary condition: $P(\theta)u = g \Rightarrow$ (*) $\partial^\alpha g(0) = 0$ if $P(\alpha) = 0$.

Assume: $P_m(e_j) \neq 0$ all j, e_j unit vectors, P has finitely many integer zeros.

Theorem: $P(\theta)u = g$ has a solution for all $g \in A(\mathbb{R}^d)$ with (*) $\Leftrightarrow P_m(z) \neq 0$ for all $z \in \mathbb{C}^d$ with $\operatorname{Re} z \in Q$.

Case of $E(\Omega) = C^\infty(\Omega)$, $\Omega = \mathbb{R}^d$

Necessary condition: $P(\theta)u = g \Rightarrow$ (**) For every $\alpha \in \mathbb{N}^J$, $J \subset D = \{1, .., d\}$: $g^{(\alpha)}(0_J, x_{D\setminus J}) = 0$ if $P(\alpha, x_{D\setminus J}) = 0$ on $\mathbb{R}^{D\setminus J}$.

Global solutions

P polynomial, P_m principal part.

Case of $E(\Omega) = A(\Omega)$, $\Omega = \mathbb{R}^d$

Necessary condition: $P(\theta)u = g \Rightarrow$ (*) $\partial^\alpha g(0) = 0$ if $P(\alpha) = 0$.

Assume: $P_m(e_j) \neq 0$ all j, e_j unit vectors, P has finitely many integer zeros.

Theorem: $P(\theta)u = g$ has a solution for all $g \in A(\mathbb{R}^d)$ with (*) $\Leftrightarrow P_m(z) \neq 0$ for all $z \in \mathbb{C}^d$ with $\mathrm{Re}\, z \in Q$.

Case of $E(\Omega) = C^\infty(\Omega)$, $\Omega = \mathbb{R}^d$

Necessary condition: $P(\theta)u = g \Rightarrow$ (**) For every $\alpha \in \mathbb{N}^J$, $J \subset D = \{1, .., d\}$: $g^{(\alpha)}(0_J, x_{D \setminus J}) = 0$ if $P(\alpha, x_{D \setminus J}) = 0$ on $\mathbb{R}^{D \setminus J}$.

Theorem: $P(\theta)u = g$ has a solution in $C^\infty(\mathbb{R}^d)$ for all $g \in C^\infty(\mathbb{R}^d)$ with (**).

Co-Authors of Paweł Domański

José Bonet
Andreas Defant
Susanne Dierolf
Lech Drewnowski
Juan Carlos Diaz Alcaide
Carmen Fernández Rosell
Leonhard Frerick
Michal Goliński
Bronisław Jakubczyk
Michal Jasiczak
Can Deha Kariksiz
Jörg Krone
Michael Langenbruch
Mikael Lindström
Mieczysław Mastyło
Jorge Mujica
Augustyn Ortyński
Melapalayam S. Ramanujan
Georg Schlüchtermann
Jari Taskinen
Dietmar Vogt
Witold Wnuk