Pointwise mutipliers of Orlicz function spaces

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The problem

$$M(L^{\varphi_1}, L^{\varphi}) = \{x \in L^0 : xy \in L^{\varphi} \text{ for every } y \in L^{\varphi_1}\} =?$$

Orlicz space

A function $\varphi : [0,\infty) \to [0,\infty]$ is called a *Young function* if it is convex, non-decreasing and $\varphi(0) = 0$.

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The *Orlicz space* L^φ is defined as

$$L^\varphi = \{x \in L^0 : I_\varphi(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where the modular I_φ is given by

$$I_\varphi(x) = \int_\Omega \varphi(|x|) d\mu$$

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where the modular I_φ is given by

$$I_\varphi(x) = \int_\Omega \varphi(|x|) d\mu$$

and the Luxemburg-Nakano norm is defined as

$$\|x\|_\varphi = \inf\{\lambda > 0 : I_\varphi(\frac{x}{\lambda}) \leqslant 1\}.$$

Pointwise multipliers

For pair of a Banach function spaces E, F over the same measure space we define the space of the *pointwise multipliers* $M(E, F)$ as

$$M(E,F) = \{x \in L^0 : xy \in F \text{ for every } y \in E\}$$

equipped with the operator norm

$$\|x\|_{M(E,F)} = \sup\{\|xy\|_F : \|y\|_E \leqslant 1\}.$$

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Example

Let $1 \leqslant q \leqslant p \leqslant \infty$ then

$$M(L^p, L^q) = L^r$$

where

$$\frac{1}{p} + \frac{1}{r} = \frac{1}{q}.$$

Remark

We have

$$M(E, L^1) = E'$$

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In particular

$$M(L^\varphi, L^1) = L^{\varphi^*}$$

where the Young conjugate φ^* of φ is defined as

$$\varphi^*(u) := \sup_{v>0}\{uv - \varphi(v)\}.$$

Pointwise product

We define the *pointwise product* of two Banach function spaces E, F as

$$E \odot F = \{z \in L^0 : z = xy \text{ for some } x \in E, y \in F\}.$$

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Ando (1960)

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Inclusion (1) is equivalent to generalized Young inequality

$$\varphi(Cuv) \leqslant \varphi_0(u) + \varphi_1(v) \tag{2}$$

for some $x_0, C > 0$ and $u, v \geqslant x_0$.

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Remark

Note that if

$$L^{\varphi_1} \odot L^{\varphi_0} \hookrightarrow L^{\varphi_1}$$

then

$$L^{\varphi_0} \hookrightarrow M(L^{\varphi_1}, L^{\varphi})$$

T. Ando *On products of Orlicz spaces*, Mathematische Annalen, 1960

O'Neil (1965)

Inequality on inverses of Orlicz functions

$$\varphi_0^{-1}(u)\varphi_1^{-1}(u) \leqslant \varphi^{-1}(u)$$

implies generalized Young inequality

$$\varphi(uv) \leqslant \varphi_0(u) + \varphi_1(v)$$

for all $u, v \geqslant 0$ thus

$$L^{\varphi_0} \hookrightarrow M(L^{\varphi_1}, L^{\varphi}).$$

R. O'Neil *Fractional Integration in Orlicz Spaces. I*, Transactions of the American Mathematical Society, 1965

Generalized inverse function

Let φ be a Young function. We define the *generalized inverse function* of φ as

$$\varphi^{-1}(v) := \inf\{u \geqslant 0 : \varphi(u) > v\}.$$

Maligranda and Persson (1989)

Let $\varphi, \varphi_0, \varphi_1$ be Orlicz functions. If

$$\varphi^{-1}(u) \leqslant \varphi_0^{-1}(u)\varphi_1^{-1}(u)$$

and generalized Young inequality holds

$$\varphi(uv) \leqslant \varphi_0(u) + \varphi_1(v)$$

then

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi_0}.$$

L. Maligranda, L. E. Persson *Generalized duality of some Banach function spaces*, Indagationes Mathematicae (Proceedings), 1989

Maligranda and Nakai (2010)

If for two given Young functions φ, φ_1 there exits third one φ_0 such that

$$B\varphi_0^{-1}(u)\varphi_1^{-1}(u) \leqslant \varphi^{-1}(u) \leqslant C\varphi_0^{-1}(u)\varphi_1^{-1}(u)$$

for some constants $B, C > 0$ and all $u > 0$ (we will write for short that $\varphi_0^{-1}\varphi_1^{-1} \approx \varphi^{-1}$) then

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi_0}$$

L. Maligranda, E. Nakai *Pointwise multipliers of Orlicz spaces*, Arch. Math., 2010

Generalized conjugate function

For two Young functions φ, φ_1 we define the conjugate of φ_1 with respect to φ as

$$\varphi \ominus \varphi_1(u) := \sup_{v>0}\{\varphi(uv) - \varphi_1(v)\}.$$

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Remark

If $\varphi(u) = u$ (and $L^\varphi = L^1$) then

$$\varphi \ominus \varphi_1(u) = \varphi_1^*(u)$$

and

$$M(L^{\varphi_1}, L^\varphi) = L^{\varphi \ominus \varphi_1}.$$

Maligranda and Persson (1989)

For two Orlicz functions φ, φ_1 if $\frac{\varphi_1}{\varphi}$ is non-decresing and

$$\limsup_{u\to\infty} \frac{\varphi(uv)}{\varphi_1(u)} = \limsup_{u\to 0} \frac{\varphi_1(u)}{\varphi(vu)} = 0$$

for any $v > 0$ then

$$\varphi^{-1}(u) \leqslant C(\varphi \ominus \varphi_1)^{-1}(u)\varphi_1^{-1}(u)$$

in consequence

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi\ominus\varphi_1}.$$

L. Maligranda, L. E. Persson *Generalized duality of some Banach function spaces*, Indagationes Mathematicae (Proceedings), 1989

Kolwicz, Leśnik, Maligranda (2013)

If $\limsup_{u\to\infty} \frac{\varphi(uv)}{\varphi_1(u)} = 0$ for any $v > 0$ and at least one of following holds

- $\frac{\varphi(uv)}{\varphi_1(u)}$ is non-increasing for any $v > 0$,
- $\frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)}$ is non-decreasing,
- $\varphi \ominus \varphi_1$ satisfies Δ_2 for large arguments

then

$$(\varphi \ominus \varphi_1)^{-1}\varphi_1^{-1} \approx \varphi^{-1} \tag{3}$$

therefore

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi\ominus\varphi_1}.$$

P. Kolwicz, K. Leśnik, L. Maligranda *Pointwise multipliers of Calderon-Lozanovskii spaces*, Math. Nachr. 2013

Question

Is equivalence

$$\varphi_0^{-1}\varphi_1^{-1} \approx \varphi^{-1} \tag{4}$$

necessary for equality

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi_0}?$$

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Kolwicz, Leśnik and Maligranda showed example of Orlicz functions φ, φ_1 such that $\varphi_0 = \varphi \ominus \varphi_1$ does not satisfy (4) but

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi \ominus \varphi_1}.$$

P. Kolwicz, K. Leśnik, L. Maligranda *Pointwise multipliers of Calderon-Lozanovskii spaces*, Math. Nachr. 2013

Djakov and Ramanujan (2000)

In sequence case we have

$$M(\ell^{\varphi_1}, \ell^{\varphi}) = \ell^{\varphi \ominus_1 \varphi_1}$$

without restrictions on Orlicz functions φ_1, φ and slightly modified definition of generalized conjugate function

$$\varphi \ominus_1 \varphi_1(u) := \sup_{0 \leqslant s \leqslant 1} \{\varphi(uv) - \varphi_1(v)\}.$$

P. B. Djakov, M. S. Ramanujan *Multipliers between Orlicz Sequence Spaces*,Turk J Math, 2000

Main result Leśnik, T. (2017)

For pair of Young functions φ_1, φ we have

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi \ominus \varphi_1}$$

K. Leśnik, J. Tomaszewski *Pointwise mutipliers of Orlicz function spaces and factorization*, Positivity, 2017

Lozanovskii theorem (1969)

Let E be a Banach function space

$$E \odot E' = E \odot M(E, L^1) = L^1.$$

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Factorization

Let E, F be function spaces. We say that E factorizes F if

$$E \odot M(E, F) = F.$$

G. Ja. Lozanovskii *On some Banach lattices*, Sibirsk. Mat. Zh., 1969

Kolwicz, Leśnik, Maligranda 2014

Let $\varphi, \varphi_0, \varphi_1$ be Young functions, then

$$L^{\varphi_0} \odot L^{\varphi_1} = L^{\varphi}$$

if, and only if

$$\varphi_1^{-1}\varphi_0^{-1} \approx \varphi^{-1}.$$

Kolwicz, Leśnik, Maligranda 2014

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Factorization of Orlicz spaces Leśnik, T. (2017)

$$L^{\varphi_1} \odot M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi}$$

if, and only if

$$\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \approx \varphi^{-1}.$$

P. Kolwicz, K. Leśnik, L. Maligranda *Pointwise products of some Banach function spaces and factorization*, J. Funct. Anal. 2014
K. Leśnik, J. Tomaszewski *Pointwise mutipliers of Orlicz function spaces and factorization*, Positivity, 2017

Thank You!