Vector measures with values in $\ell^\infty(\Gamma)$ and interpolation of Banach lattices

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Motivation

We shall characterize the space $\left(L^p(\mu_0), L^p(\mu_1)\right)_{\theta,p}$ where μ_0 and μ_1 are two positive measures. We may assume that μ_0 and μ_1 are absolutely continuous with respect to a third measure μ. Thus we suppose that

$$d\mu_0 = w_0 d\mu \quad \text{and} \quad d\mu_1 = w_1 d\mu$$

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$$d\mu_0 = w_0 d\mu \quad \text{and} \quad d\mu_1 = w_1 d\mu$$

Theorem [1958 Stein, Weiss]

Let μ be a positive scalar measure. If $0 < p \leqslant \infty$ and $0 < \theta < 1$, then

$$\left(L^p(w_0 d\mu), L^p(w_1 d\mu)\right)_{\theta,p} = L^p(w_0^{1-\theta} w_1^{\theta} d\mu)$$

E. M. Stein, G. Weiss
Interpolation of operators with change of measures
Trans. Amer. Math. Soc. 87 (1958), 159–172

Lattices and Couples

Banach lattices

Definition

We call X a Banach lattice if

- $X \subset L^0(\Omega, \Sigma, \mu)$ – the space of complex valued measurable functions on Ω. The order $|f| \leqslant |g|$ means $|f(\omega)| \leqslant |g(\omega)|$ for μ-almost all $\omega \in \Omega$
- $|f| \leqslant |g|$ with $f \in L^0(\Omega)$ and $g \in X$ implies $f \in X$ with

$$\|f\|_X \leqslant \|g\|_X$$

- X is complete and it contains the collection $\mathrm{sim}\Sigma$ of all Σ-simple functions

The role model for X are Lebesgue spaces, other important examples are

- Lorentz spaces
- Orlicz spaces
- Marcinkiewicz spaces

Banach couple

Definition

We call $\vec{A} := (A_0, A_1)$ a Banach couple if both A_0 and A_1 are Banach spaces such that

$$A_0, A_1 \hookrightarrow \mathcal{X}$$

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For a given Banach couple $\vec{A}$, we define spaces

- intersection $A_0 \cap A_1$ with the norm

$$\|a\|_{A_0 \cap A_1} = \max\left\{\|a\|_{A_0}, \|a\|_{A_1}\right\}$$

- sum $A_0 + A_1$ with the norm

$$\|a\|_{A_0 + A_1} = \inf_{a = a_0 + a_1}\left\{\|a_0\|_{A_0} + \|a_1\|_{A_1}\right\}$$

Definition

We say that $\vec{A}$ is regular if $A_0 \cap A_1$ is dense in in both A_0 and A_1

Peetre's K–functional

- $\vec{A} = (A_0, A_1)$ — a Banach couple.

Definition

The K-functional of $\vec{A}$ is the map from the sum $A_0 + A_1$ into the cone of the nonnegative concave functions defined on $(0, \infty)$, given by

$$K(t, a) = K(t, a; \vec{A}) := \inf_{a=a_0+a_1} \{ \|a_0\|_{A_0} + t \|a_1\|_{A_1} \}, \quad t > 0$$

- $K(t, a)$ is an equivalent norm on the sum $A_0 + A_1$

Couples of Banach lattices

- Any two Banach lattices Y_0 and Y_1 based on (Ω, Σ, μ) form a Banach couple $\vec{Y}$

$$Y_0, Y_1 \hookrightarrow L^0(\mu)$$

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Definition

- Given a Banach lattice $(Y, \|\cdot\|_Y)$, the order continuous (absolutely continuous) part Y_a of Y is defined by

$$Y_a := \Big\{ f \in Y \,:\, |f| \geqslant |f_n| \downarrow 0 \quad \text{with} \quad f_n \in Y \quad \Longrightarrow \quad \|f_n\|_Y \downarrow 0 \Big\}$$

- The Banach lattice Y is said to be order continuous if $Y = Y_a$

Examples

L^p spaces are order continuous for $p < \infty$ and $(L^\infty)_a = \{0\}$

- If Y_0 and Y_1 are order continuous, then $Y_0 \cap Y_1$ is dense in both Y_0 and Y_1 — the couple $\vec{Y}$ is regular

Vector measure toolbox

Vector measure

- (Ω, Σ) — a measurable space, X — a Banach space
- We call $m \colon \Sigma \to X$ a vector measure if m is a σ-additive set function

Theorem [1929 Orlicz, 1938 Pettis]

A Banach space valued, finitely additive set function $m \colon \Sigma \to X$ is σ-additive if and only if it is scalarly σ-additive

$$\langle m, x^* \rangle \quad \textit{is} \quad \sigma - \textit{additive for every} \quad x^* \in X^*$$

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Definition

- $\mathcal{L}^0(\Sigma)$ — the space of all $\mathbb{C}$-valued, Σ-measurable functions on Ω
- $f \in \mathcal{L}^0(\Sigma)$ is called m-integrable if
  (1) it is $\langle m, x^* \rangle$-integrable for every $x^* \in X^*$
  (2) for every $A \in \Sigma$ there exists a unique element $m_f(A) \in X$ satisfying

$$\langle m_f(A), x^* \rangle = \int_A f \, d\langle m, x^* \rangle, \quad x^* \in X^*$$

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Definition

- m_f is called the indefinite integral of f relative to m

$$\int_A f \, dm := m_f(A), \quad A \in \Sigma$$

- $L^1(m)$ denote the space of all m-integrable functions on Ω, equipped with the seminorm $\| \cdot \|_{L^1(m)}$ defined by

$$\|f\|_{L^1(m)} := \sup_{x^* \in B_{X^*}} \int_\Omega |f| \, d|\langle m, x^* \rangle|, \quad f \in L^1(m)$$

L^1 of a vector measure m

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Each Σ-simple function is m-integrable

$$f = \sum_i \lambda_i \chi_{A_i}, \quad \text{with} \quad \{A_i\} \quad \text{disjoint.}$$

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Definition

- Every function $f \in L^1(m)$ satisfying $\|f\|_{L^1(m)} = 0$ is called m-null
- $\mathcal{N}(m)$ — the subspace of $L^1(m)$ consisting of all m-null functions

- We identify $L^1(m)$ with its quotient space $L^1(m)/\mathcal{N}(m)$

Rybakov control measure for m

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Definition

- We call a finite measure $\mu\colon \Sigma \to [0, \infty)$ a control measure for $m\colon \Sigma \to X$ if μ and m are mutually absolutely continuous

$$\mathcal{N}_0(\mu) = \mathcal{N}_0(m)$$

Theorem [1970 Rybakov]

Let $m\colon \Sigma \to X$ be a vector measure, then there exists a linear functional $x^* \in X^*$ such that the scalar measure $|\langle m, x^* \rangle|\colon \Sigma \to [0, \infty)$ is a control measure for m

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- Given any control measure μ for m — the space $L^1(m) \subset L^0(\mu)$ is an order continuous Banach lattice based on the measure space (Ω, Σ, μ)

$L^1(m)$ is an order continuous Banach lattice

Theorem [1970 Lewis]

A function $f \in \mathcal{L}^0(\Sigma)$ is m-integrable $\iff$ there exists a sequence $\{s_n\}_{n=1}^{\infty} \subset \mathrm{sim}\Sigma$ such that $s_n \to f$ pointwise as $n \to \infty$ and such that the sequence $\{\int_A s_n \, dm\}_{n=1}^{\infty}$ converges in X for every $A \in \Sigma$ with

$$\int_A f \, dm = \lim_{n \to \infty} \int_A s_n \, dm$$

- $L^1(m)$ — an order continuous Banach lattice — $\mathrm{sim}\Sigma$ is dense in it
- Every order continuous Banach lattice can be realized as $L^1(m)$ for a suitable vector measure m

Interpolation of vector measures

Calderón's interpolation method

- Given any pair of Banach lattices Y_0, Y_1 on (Ω, Σ, μ), Calderón constructs the Banach lattice

$$Y_\theta = Y_0^{1-\theta} Y_1^\theta$$

to consists of all $f \in L^0(\mu)$ satisfying $|f| \leqslant |f_0|^{1-\theta}|f_1|^\theta$ μ-a.e. for some $f_0 \in Y_0$ and $f_1 \in Y_1$

- Y_θ is equipped with the norm

$$\|f\|_{Y_\theta} := \inf_{|f| \leqslant |f_0|^{1-\theta}|f_1|^\theta} \|f_0\|_{Y_0}^{1-\theta} \|f_1\|_{Y_1}^\theta$$

where the infimum runs over all possible representations of f.

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  where the infimum runs over all possible representations of f.
- If either Y_0 or Y_1 is order continuous then Y_θ is also order continuous and the equality

$$Y_0^{1-\theta} Y_1^\theta = [Y_0, Y_1]_\theta$$

  holds with equality of norms where $[Y_0, Y_1]_\theta$ is the lower complex interpolation space

Previously on Interpolated Measures

- Let $0 < \theta < 1$ and take two positive vector measures $m_0 \colon \Sigma \to X_0$ and $m_1 \colon \Sigma \to X_1$ on the same measurable space (Ω, Σ) such that the Calderón lattice interpolation space $X_0^{1-\theta} X_1^{\theta}$ is order-continuous
- Let $A \in \Sigma$ and $\pi \in \Pi(\Omega)$ be the set of finite measurable partitions of Ω

Definition

The interpolated vector measure

$$[m_0, m_1]_\theta = \inf_{\pi \in \Pi(\Omega)} \sum_{B \in \pi} m_0(A \cap B)^{1-\theta} m_1(A \cap B)^{\theta}$$

R. del Campo, A. Fernández, F. Mayoral, F. Naranjo, E. A. Sánchez-Pérez
Interpolation of vector measures
Acta Math. Sinica 27(1) (2011) 119–134

The nonexistence of Radon–Nikodým derivatives

Example

$\Omega = [0,1]$. $([0,1], \mathcal{M}, \lambda)$ — a Lebesgue measure space. Consider the vector measures

$$m_0(A)\colon A \in \mathcal{M} \longmapsto \big(\lambda(A), 0\big) \in \mathbb{R}^2$$
$$m_1(A)\colon A \in \mathcal{M} \longmapsto \big(0, \lambda(A)\big) \in \mathbb{R}^2$$

- $L^1(m_0 + m_1) \cong L^1(\Omega)$ and
$$\int_\Omega f \, d\big(m_0 + m_1\big) = \left(\int_\Omega f \, dm_0, \int_\Omega f \, dm_1\right)$$
- $m_0(\Omega) = (1,0)$ and $m_1(\Omega) = (0,1)$.
- There are no functions $f_0, f_1 \in L^1(m_0 + m_1)$ such that
$$\int_\Omega f_0 \, d(m_0 + m_1) = m_0(\Omega) \quad \text{and} \quad \int_\Omega f_1 \, d(m_0 + m_1) = m_1(\Omega)$$

Calderón's interpolation method

Radon–Nikodým derivatives

- Let μ_0 and μ_1 be two Rybakov measures for m_0 and m_1, respectively
- Take $\mu = (\mu_0 + \mu_1)/2$.

Definition

Define the sets

$$N_0 := \left\{ h_\phi := \frac{d|\langle m_0, \phi\rangle|}{d\mu}, \quad \phi \in \Lambda_0 := B_{X_0^*} \right\}$$

$$N_1 := \left\{ g_\psi := \frac{d|\langle m_1, \psi\rangle|}{d\mu}, \quad \psi \in \Lambda_1 := B_{X_1^*} \right\}$$

The space $\ell^\infty(\Lambda, L^1(\mu))$

$$N_0 := \left\{ h_\phi := \frac{d|\langle m_0, \phi\rangle|}{d\mu}, \quad \phi \in \Lambda_0 := B_{X_0^*} \right\}$$

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Theorem

$$I_0, I_1 : \ell^\infty(\Lambda_0, L^1(\mu)) \to \ell^\infty(\Lambda_0 \times \Lambda_1, L^1(\mu))$$

given by

$$I_0\big((h_\phi)_{\phi\in\Lambda_0}\big) := (h_{\phi,\psi})_{(\phi,\psi)\in\Lambda_0\times\Lambda_1} \quad (\text{for} \quad h_{\phi,\psi} := h_\phi) \qquad I_1\big((g_\psi)_{\phi\in\Lambda_1}\big) := (g_{\phi,\psi})_{(\phi,\psi)\in\Lambda_0\times\Lambda_1} \quad (\text{for} \quad g_{\phi,\psi} := g_\psi)$$

The mappings I_0 and I_1 are isometric embeddings

The vector measures $\overline{m}_0$ and $\overline{m}_1$

Definition

Consider the set functions

$$\overline{m}_0, \overline{m}_1 \colon \Sigma \to \ell^\infty(\Lambda_0 \times \Lambda_1, L^1(\mu))$$

given by

$$\overline{m}_0(A) := (h_{\phi,\psi} \cdot \chi_A)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1} \qquad \overline{m}_1(A) := (g_{\phi,\psi} \cdot \chi_A)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}$$
$$(\text{for} \quad h_{\phi,\psi} := h_\phi) \qquad (\text{for} \quad g_{\phi,\psi} := g_\psi)$$

Both of them define vector measures

- $L^1(m_0) \cong L^1(\overline{m}_0)$ and $L^1(m_1) \cong L^1(\overline{m}_1)$ — both spaces are isometrically order isomorphic

The θ-interpolated vector measure

Definition

Given two vector measures as $\overline{m}_0$ and $\overline{m}_1$, we define the θ-interpolated vector measure

$$\overline{m}_\theta \colon \Sigma \to \ell^\infty(\Lambda_0 \times \Lambda_1, L^1(\mu))$$

by

$$\overline{m}_\theta(A) := (h_\phi^{1-\theta} \cdot g_\psi^\theta \cdot \chi_A)_{(\phi,\psi)\in\boldsymbol{\Lambda_0}\times\boldsymbol{\Lambda_1}}$$

for $A \in \Sigma$

Theorem [2018 Sánchez Pérez, Szwedek]

Let $m_0 \colon \Sigma \to X_0$ and $m_1 \colon \Sigma \to X_1$ be Banach space valued measures. The equality hold isometrically and in the order

$$\big(L^1(m_0)\big)^{1-\theta}\big(L^1(m_1)\big)^\theta \cong L^1(\overline{m}_\theta)$$

The real method of interpolation

Continuous variants of the abstract real interpolation method

$$K(t,a) = K(t,a;\vec{A}) := \inf_{a=a_0+a_1} \left\{ \|a_0\|_{A_0} + t\,\|a_1\|_{A_1} \right\}, \qquad t > 0$$

- Let E be a lattice over a measurable space $(\mathbb{R}_+, dt/t)$, satisfying the condition $1 \wedge t \in E$
- The K-method space $\vec{A}_{E;K} := (A_0, A_1)_{E;K}$ consists of all elements $a \in A_0 + A_1$ such that $K(\,\cdot\,, a; \vec{A}) \in E$ and equipped with the norm

$$\|a\| := \left\| K(\,\cdot\,, a; \vec{A}) \right\|_E$$

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- In case where $E = L^q(t^{-\theta}, dt/t)$
  (1) $1 \leqslant q < \infty$ and $\theta \in (0,1)$
  (2) $q = \infty$ and $\theta \in [0,1]$
  we obtain the classical real interpolation spaces $\vec{A}_{\theta,q}$

$$(A_0, A_1)_{E;K} = (A_0, A_1)_{\theta,q}$$

You have seen this slide before!

- X_0 and X_1 — Banach lattices
- $m_0 \colon \Sigma \to X_0$ and $m_1 \colon \Sigma \to X_1$ — vector measures
- μ — the mean of two Rybakov measures μ_0, μ_1

$$N_0 := \left\{ h_\phi := \frac{d|\langle m_0, \phi \rangle|}{d\mu}, \quad \phi \in \Lambda_0 := B_{X_0^*} \right\}$$

$$N_1 := \left\{ g_\psi := \frac{d|\langle m_1, \psi \rangle|}{d\mu}, \quad \psi \in \Lambda_1 := B_{X_1^*} \right\}$$

The vector measures $\widehat{m}_0$ and $\widehat{m}_1$

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Definition

We define $\widehat{m}_0 \colon \Sigma \to \ell^\infty(\Lambda_0 \times \Lambda_1)$ and $\widehat{m}_1 \colon \Sigma \to \ell^\infty(\Lambda_0 \times \Lambda_1)$ by

$$\widehat{m}_0(A) := \left(\int_A h_{\phi,\psi} \, d\mu \right)_{(\phi,\psi) \in \Lambda_0 \times \Lambda_1}, \quad h_{\phi,\psi} = h_\phi$$

and

$$\widehat{m}_1(A) := \left(\int_A g_{\phi,\psi} \, d\mu \right)_{(\phi,\psi) \in \Lambda_0 \times \Lambda_1}, \quad g_{\phi,\psi} = g_\psi$$

for $A \in \Sigma$

The space $\ell^\infty(\Lambda_0 \times \Lambda_1)$

Definition

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for $A \in \Sigma$

Definition

Define a new set function $\Sigma(\widehat{m}_0, \widehat{m}_1) \colon \Sigma \to \ell^\infty(\Lambda_0 \times \Lambda_1)$ by

$$\Sigma(\widehat{m}_0, \widehat{m}_1)(A) := \Big(\int_A \min\{h_\phi, g_\psi\}\, d\mu\Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}$$

The key lemma

Lemma [2018 Sánchez Pérez, Szwedek]

For all functions $f \in L^1(m_0) + L^1(m_1)$ *and* $t > 0$, *the following equality holds*

$$K\Big(t, f; \big(L^1(m_0), L^1(m_1)\big)\Big) = \sup_{(\phi,\psi)\in\Lambda_0\times\Lambda_1} K\Big(t, f; \big(L^1(h_\phi d\mu), L^1(g_\psi d\mu)\big)\Big)$$

- $L^1(\Sigma(\widehat{m}_0, \widehat{m}_1)) \cong L^1(m_0) + L^1(m_1)$

ℓ^∞-valued version of the K-functional

Definition

We write

$$\widehat{K}(t,f)\colon (0,\infty)\times L^1\big(\Sigma(\widehat{m}_0,\widehat{m}_1)\big)\to \ell^\infty(\Lambda_0\times\Lambda_1)$$

for the function

$$\widehat{K}(t,f) := \Big(\int |f|\min\{h_\phi, t\,g_\psi\}d\mu\Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}$$

By the key lemma, we have

$$\widehat{K}(t,f) = \Big(K\Big(t,f;\big(L^1(h_\phi d\mu), L^1(g_\psi d\mu)\big)\Big)\Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}$$

and

$$\big\|\widehat{K}(t,f)\big\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} = K\Big(t,f;\big(L^1(m_0),L^1(m_1)\big)\Big)$$

The steps towards the interpolated measure

- Let us write $\Sigma_t(\widehat{m}_0, \widehat{m}_1)$ for the vector measure given by $\widehat{K}(t, \chi)$

$$\Sigma_t(\widehat{m}_0, \widehat{m}_1)(A) = \Big(\int_A \min\big\{ h_\phi, t\, g_\psi \big\} d\mu \Big)_{(\phi,\psi)\in\Lambda_0\times\Lambda_1}$$

- Consider the vector space

$$\mathcal{F}\big((0,\infty); \ell^\infty(\Lambda_0 \times \Lambda_1)\big)$$

of all the $\ell^\infty(\Lambda_0 \times \Lambda_1)$-valued functions

- Take a function

$$(\widehat{m}_0, \widehat{m}_1)_{\mathcal{F},\widehat{K}} \colon \Sigma \to \mathcal{F}\big((0,\infty); \ell^\infty(\Lambda_0 \times \Lambda_1)\big)$$

given by

$$(\widehat{m}_0, \widehat{m}_1)_{\mathcal{F},\widehat{K}}(A) := \int_A d\,\Sigma_t(\widehat{m}_0, \widehat{m}_1)$$

The steps towards the interpolated measure (cont.)

- The vector space
$$\mathcal{F}\big((0,\infty);\ell^\infty(\Lambda_0\times\Lambda_1)\big)$$
of all the $\ell^\infty(\Lambda_0\times\Lambda_1)$-valued functions
- The function
$$(\widehat{m}_0,\widehat{m}_1)_{\mathcal{F},\widehat{K}}(A) := \int_A d\,\Sigma_t(\widehat{m}_0,\widehat{m}_1)$$
is a countably additive, positive set function with
$$(\widehat{m}_0,\widehat{m}_1)_{\mathcal{F},\widehat{K}}(A)(t) = \Sigma_t(\widehat{m}_0,\widehat{m}_1)(A) = \Big(\int_A \min\big\{h_\phi, t\,g_\psi\big\}d\mu\Big)_{(\phi,\psi)\in\boldsymbol{\Lambda_0}\times\boldsymbol{\Lambda_1}}$$

Stay motivated !

The interpolated measure

- E — a Banach lattice over a measurable space $(\mathbb{R}_+, dt/t)$ satisfying $1 \wedge t \in E$
- We denote by $E\big(\ell^\infty(\Lambda_0 \times \Lambda_1)\big)$ the Köthe-Bochner space of all

$$\mathcal{G} \in \mathcal{F} := \mathcal{F}\big((0,\infty); \ell^\infty(\Lambda_0 \times \Lambda_1)\big)$$

equipped with the norm

$$\|\mathcal{G}\|_{E(\ell^\infty(\Lambda_0\times\Lambda_1))} := \Big\| \|\mathcal{G}(\,\cdot\,)\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} \Big\|_E$$

$$E\big(\ell^\infty(\Lambda_0 \times \Lambda_1)\big) := \Big\{ \mathcal{G} \in \mathcal{F} \,:\, \Big\| \|\mathcal{G}(\,\cdot\,)\|_{\ell^\infty(\Lambda_0\times\Lambda_1)} \Big\|_E < \infty \Big\}$$

The interpolated measure

- E — a Banach lattice over a measurable space $(\mathbb{R}_+, dt/t)$ satisfying $1 \wedge t \in E$
- We denote by $E\big(\ell^\infty(\Lambda_0 \times \Lambda_1)\big)$ the Köthe-Bochner space of all

$$\mathcal{G} \in \mathcal{F} := \mathcal{F}\big((0,\infty); \ell^\infty(\Lambda_0 \times \Lambda_1)\big)$$

equipped with the norm

$$\|\mathcal{G}\|_{E(\ell^\infty(\Lambda_0 \times \Lambda_1))} := \Big\| \|\mathcal{G}(\,\cdot\,)\|_{\ell^\infty(\Lambda_0 \times \Lambda_1)} \Big\|_E$$

$$E\big(\ell^\infty(\Lambda_0 \times \Lambda_1)\big) := \Big\{ \mathcal{G} \in \mathcal{F} \,:\, \Big\| \|\mathcal{G}(\,\cdot\,)\|_{\ell^\infty(\Lambda_0 \times \Lambda_1)} \Big\|_E < \infty \Big\}$$

Definition

We write in this case $(\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}$ for the set function $(\widehat{m}_0, \widehat{m}_1)_{\mathcal{F};\widehat{K}}$

- The set function $(\widehat{m}_0, \widehat{m}_1)_{E;\widehat{K}}$ is a positive vector measure on $E\big(\ell^\infty(\Lambda_0 \times \Lambda_1)\big)$

Interpolation of vector measures by the real method

Theorem [2018 Sánchez Pérez, Szwedek]

- *Suppose that both X_0 and X_1 are Banach lattices*
- *Let $m_0 \colon \Sigma \to X_0$ and $m_1 \colon \Sigma \to X_1$ be two arbitrary vector measures*
- *Let E be a Banach lattice of functions on $(\mathbb{R}_+, dt/t)$ satisfying the condition $1 \wedge t \in E$*

Then

$$\Big(\big(L^1(m_0), L^1(m_1) \big)_{E;K} \Big)_a \cong L^1\Big(\big(\widehat{m}_0, \widehat{m}_1 \big)_{E;\widehat{K}} \Big)$$

E. A. Sánchez Pérez, R. Szwedek

Vector measures with values in $\ell^\infty(\Gamma)$ and interpolation of Banach lattices

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Classical cases

- In the case where $E = L^q(t^{-\theta}, dt/t)$, with $1 \leqslant q < \infty$ and $\theta \in (0,1)$ we have

Corollary

If $1 \leqslant q < \infty$ *and* $0 < \theta < 1$, *then*

$$(L^1(m_0), L^1(m_1))_{\theta,q} \cong L^1\left((\widehat{m}_0, \widehat{m}_1)_{L^q(t^{-\theta},\, dt/t);\widehat{K}}\right)$$

- In the case where $E = L^\infty(t^{-\theta}, dt/t)$ with $\theta \in [0,1]$ we have

Corollary

If $0 \leqslant \theta \leqslant 1$, *then*

$$\left((L^1(m_0), L^1(m_1))_{\theta,\infty}\right)_a \cong L^1\left((\widehat{m}_0, \widehat{m}_1)_{F;\widehat{K}}\right)$$

for

$$F = \overline{L^\infty \cap L^\infty(t^{-1})}^{L^\infty(t^{-\theta},dt/t)}$$

Thank you for your attention!

Any questions?