Bishop-Phelps-Bollobás moduli for operators

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joint work with Vladimir Kadets

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Norm-attaining functionals

Let X be a real Banach spaces X^* be its dual space.
Recall that a functional $x^* \in X^*$ ***attains its norm***, if there is an $x \in S_X$ with $x^*(x) = \|x^*\|$. The famous Bishop-Phelps theorem (1961) says that for every Banach space X **the subset of norm-attaining functionals is dense in** X^*.

Theorem (Bishop-Phelps-Bollobás theorem, 1970)

Let X be a Banach space. Suppose $x \in S_X$ and $x^* \in S_{X^*}$ satisfy $x^*(x) \geq 1 - \varepsilon$ $(0 < \varepsilon < 1/8)$. Then there exists $(y, y^*) \in S_X \times S_{X^*}$ with $y^*(y) = 1$, such that $\|x - y\| < \sqrt{2\varepsilon} + 2\varepsilon$ and $\|x^* - y^*\| \leq \sqrt{2\varepsilon}$.

We use the notation:

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\},$$
$$\Pi_\varepsilon(X) = \{(x, x^*) \in B_X \times B_{X^*} : x^*(x) \geq 1 - \varepsilon\}.$$

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M. Chica, V. Kadets, M. Martin, S. Moreno-Pulido, and F. Rambla-Barreno in the paper "Bishop-Phelps-Bollobás moduli of a Banach space"(2014) introduced the following definition:

Definition

The Bishop-Phelps-Bollobás modulus of X is the function $\Phi_X(\varepsilon) : (0, 2) \longrightarrow \mathbb{R}^+$:

$$\boldsymbol{\Phi_X(\varepsilon)} = \inf\{\delta > 0 : \ \forall\, (x, x^*) \in \Pi_\varepsilon(X) \quad \exists\, (y, y^*) \in \Pi(X) \text{ such that } \|x - y\| < \delta, \|x^* - y^*\| < \delta\}.$$

Some properties of Φ_X

$$\Phi_X(\varepsilon) = \inf\{\delta > 0 : \ \forall\, (x, x^*) \in \Pi_\varepsilon(X) \quad \exists\, (y, y^*) \in \Pi(X) \text{ such that } \|x - y\| < \delta, \|x^* - y^*\| < \delta\}.$$

1. $\Phi_X(\varepsilon)$ is a continuous non-decreasing function.
2. $\Phi_X(\varepsilon) \leq \Phi_{X^*}(\varepsilon)$.
3. $\Phi_X(\cdot)$ is continuous with respect to X.
4. $\Phi_X(\varepsilon) \leq \sqrt{2\varepsilon}$.

When instead of functionals one considers **operators**, the situation changes: the set of norm-attaining operators **is not always dense** in the space $L(X, Y)$ of all linear operators acting from X to Y. (Lindenstrauss, 1963).

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Definition (Acosta, Aron, García and Maestre, 2008)

A couple of Banach spaces (X, Y) is said to have ***the Bishop-Phelps-Bollobás property for operators*** if for any $\delta > 0$ there exists $\varepsilon(\delta) > 0$, such that for every operator $T \in S_{L(X,Y)}$, if $x \in S_X$ and $\|T(x)\| > 1 - \varepsilon(\delta)$, then there exist $y \in S_X$ and $F \in S_{L(X,Y)}$ satisfying

$$\|F(y)\| = 1, \|x - y\| < \delta \text{ and } \|T - F\| < \delta.$$

In our work we studied the possible estimates from above and from below for parameters that measure the rate of approximation in the Bishop-Phelps-Bollobás property for operators.

$$\Pi(X,Y) = \left\{(x,T) \in S_X \times S_{L(X,Y)} : \|T(x)\| = 1\right\},$$
$$\Pi_\varepsilon(X,Y) = \left\{(x,T) \in B_X \times B_{L(X,Y)} : \|T(x)\| \geq 1-\varepsilon\right\}.$$

Definition

The Bishop-Phelps-Bollobás modulus for operators:

$$\mathbf{\Phi}_{(X,Y)}(\varepsilon) = \inf\{\delta > 0 : \forall\ (x,T) \in \Pi_\varepsilon(X,Y) \quad \exists\ (y,F) \in \Pi(X,Y) \text{ such that } \|x-y\| < \delta, \|T-F\| < \delta\}.$$

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Remark

(X,Y) has the BPB property for operators $\Leftrightarrow$ $\Phi_{(X,Y)}(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$

Property β

Definition (Lindenstrauss, 1963)

A Banach space Y has ***the property*** β if there are two sets $\{y_\alpha : \alpha \in \Lambda\} \subset S_Y$, $\{y^*_\alpha : \alpha \in \Lambda\} \subset S^*_Y$ and $0 \leq \rho < 1$ such that

$$y^*_\alpha(y_\alpha) = 1,$$

$$|y^*_\alpha(y_\gamma)| \leq \rho \quad \text{if} \quad \alpha \neq \gamma, \text{ and}$$

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$\|y\| = \sup\{|y^*_\alpha(y)| : \alpha \in \Lambda\}$ for all $y \in Y$.

Examples

1. Polyhedral finite-dimensional spaces have this property.
2. c_0 and ℓ_∞ also have this property with $\rho = 0$.

Theorem (Acosta, Aron, García and Maestre, 2008)

Let X and Y be Banach spaces and Y has the property β. Then the pair (X, Y) has the BPB property for operators.

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Theorem (Kadets, S., 2017)

Let X and Y be Banach spaces and Y has the property β. Then

$$\Phi_{(X,Y)}(\varepsilon) \le \min\left\{\sqrt{2\varepsilon}\sqrt{\frac{1+\rho}{1-\rho}},\ 2\right\}, \quad \varepsilon \in (0,1).$$

The accuracy of estimation

$$\Phi_{(X,Y)}(\varepsilon) \leq \min\left\{\sqrt{2\varepsilon}\sqrt{\frac{1+\rho}{1-\rho}},\ 2\right\}, \quad \varepsilon \in (0,1).$$

1. If Y has the property β with $\rho = 0$, then

$$\Phi_{(\ell_1^{(2)},Y)}(\varepsilon) = \min\left\{\sqrt{2\varepsilon},\ 1\right\}, \quad \varepsilon \in (0,1).$$

The accuracy of estimation

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1. If Y has the property β with $\rho = 0$, then

$$\Phi_{(\ell_1^{(2)},Y)}(\varepsilon) = \min\left\{\sqrt{2\varepsilon},\ 1\right\}, \quad \varepsilon \in (0,1).$$

2. For $\rho \in [1/2, 1)$ there is the space $Y = Y_\rho$ such that

$$\Phi_{(\ell_1^{(2)},Y)}(\varepsilon) \geq \min\left\{\sqrt{2\varepsilon}\sqrt{\frac{\rho}{1-\rho}},\ 1\right\}, \quad \varepsilon \in (0,1).$$

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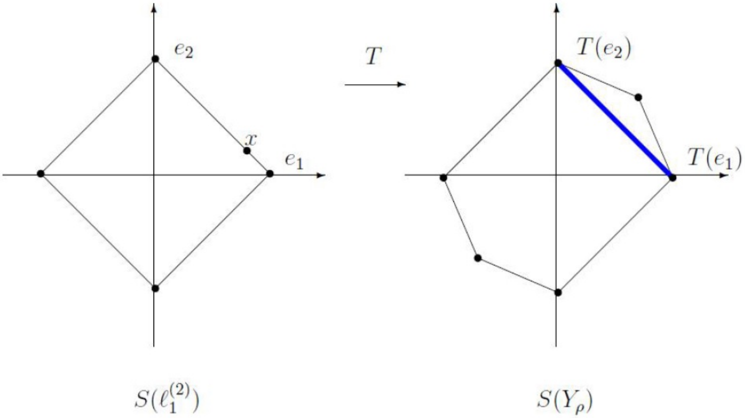

Non-continuity of the modulus

Theorem (Kadets, S., 2017)

$\Phi_{(X,Y)}(\varepsilon)$ as a function of Y is not continuous in the sense of Banach-Mazur distance. Namely, for every $\varepsilon \in (0, 1/2)$
$\Phi_{(\ell_1^{(2)}, Y_\rho)}(\varepsilon) \underset{\rho \to 1}{\not\to} \Phi_{(\ell_1^{(2)}, \ell_1^{(2)})}(\varepsilon)$.

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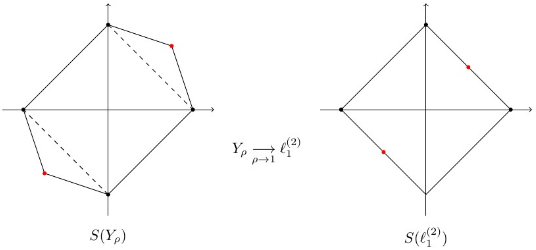
$$\Phi_{(\ell_1^{(2)}, Y_\rho)}(\varepsilon) \geq \min\left\{\sqrt{\frac{2\rho\varepsilon}{1-\rho}}, 1\right\} \underset{\rho\to 1}{\longrightarrow} 1, \qquad \Phi_{(\ell_1^{(2)}, \ell_1^{(2)})}(\varepsilon) = \min\{\sqrt{2\varepsilon}, 1\}$$



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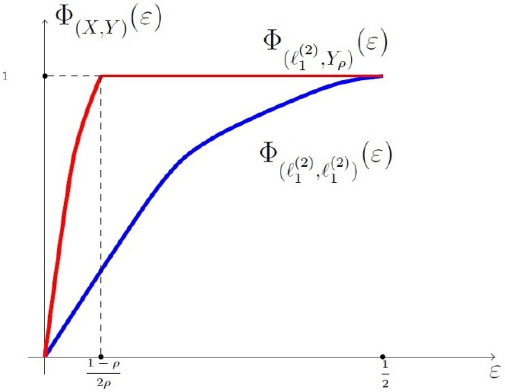

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