Operator calculus for noncommuting operators over symmetric Fock space

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Let us fix any real $\beta > 1$. An infinitely differentiable function φ is called to be a Gevrey ultradifferentiable if for each segment $[\mu, \nu] \subset \mathbb{R}$ there exist constants $h > 0$ and $C > 0$ such that the inequality $\sup_{t\in[\mu,\nu]} |\partial^k \varphi(t)| \le C h^k k^{k\beta}$ holds for all $k \in \mathbb{Z}_+$. For a fixed $h > 0$ let us consider the subspace

$$\mathcal{G}_\beta^h[\mu,\nu] := \big\{\varphi \in C^\infty : \operatorname{supp}\varphi \subset [\mu,\nu], \|\varphi\|_{\mathcal{G}_\beta^h[\mu,\nu]} < \infty\big\},$$

where $\|\varphi\|_{\mathcal{G}_\beta^h[\mu,\nu]} := \sup_{k\in\mathbb{Z}_+} \sup_{t\in[\mu,\nu]} \frac{|\partial^k \varphi(t)|}{h^k k^{k\beta}}$.

Consider the space

$$\mathcal{G}_\beta := \bigcup_{\mu<\nu,\, h>0} \mathcal{G}_\beta^h[\mu,\nu], \qquad \mathcal{G}_\beta \simeq \operatorname*{ind\,lim}_{\mu<\nu,\, h>0} \mathcal{G}_\beta^h[\mu,\nu],$$

of Gevrey ultradifferentiable functions with compact supports and endow it with topology of inductive limit. Let $\mathcal{G}'_\beta$ be its dual space of Roumieu ultradistributions.

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Let $h > 0$ be any positive real and $\mu, \nu \in \mathbb{R}$ be any reals such that $\mu < \nu$. In the space of entire functions of exponential type we consider the subspace $E^h_\beta[\mu,\nu]$ of functions with the finite norm

$$\|\psi\|_{E^h_\beta[\mu,\nu]} := \sup_{k\in\mathbb{Z}_+} \sup_{z\in\mathbb{C}} \frac{|z^k \psi(z) e^{-H_{[\mu,\nu]}(\eta)}|}{h^k k^{k\beta}},$$

where $\quad H_{[\mu,\nu]}(\eta) := \sup_{t\in[\mu,\nu]} t\eta.$

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Consider the Fourier-Laplace transformation

$$\hat{\varphi}(z) := (F\varphi)(z) = \int_{\mathbb{R}} e^{-itz}\varphi(t)\,dt, \qquad \varphi \in \mathcal{G}_\beta,\ z \in \mathbb{C}.$$

It has been shown, that $F(\mathcal{G}_\beta) = E_\beta$.

For any locally convex (l.c.) space $\mathcal{X}$, let $\mathcal{X}^{\hat{\otimes} n}$, $n \in \mathbb{N}$, be the symmetric nth tensor degree of $\mathcal{X}$, completed in the projective tensor topology.

To define the l.c. space $\mathcal{P}({}^n\mathcal{G}'_\beta)$ of n-homogeneous polynomials on $\mathcal{G}'_\beta$ we use the isomorphism $\mathcal{P}({}^n\mathcal{G}'_\beta) \simeq (\mathcal{G}'^{\hat{\otimes} n}_\beta)'$, described in [1]. We equip $\mathcal{P}({}^n\mathcal{G}'_\beta)$ with the locally convex topology $\mathfrak{b}$ of uniform convergence on bounded sets in $\mathcal{G}'_\beta$. Set $\mathcal{P}({}^0\mathcal{G}'_\beta) := \mathbb{C}$. The space $\mathcal{P}(\mathcal{G}'_\beta)$ of all continuous polynomials on $\mathcal{G}'_\beta$ is defined to be the complex linear span of all $\mathcal{P}({}^n\mathcal{G}'_\beta)$, $n \in \mathbb{Z}_+$, endowed with the topology $\mathfrak{b}$. Let $\mathcal{P}'(\mathcal{G}'_\beta)$ mean the strong dual of $\mathcal{P}(\mathcal{G}'_\beta)$.
Elements of the spaces $\mathcal{P}(\mathcal{G}'_\beta)$ and $\mathcal{P}'(\mathcal{G}'_\beta)$ we call the polynomial test ultradifferentiable functions and polynomial ultradistributions, respectively.

[1] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1999.

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Denote $\Gamma(\mathcal{G}_\beta) := \bigoplus_{n\in\mathbb{Z}_+}{}_{fin}\, \mathcal{G}_\beta^{\hat{\otimes} n}$ and $\Gamma(\mathcal{G}'_\beta) := \times_{n\in\mathbb{Z}_+} \mathcal{G}'^{\hat{\otimes} n}_\beta$. In what follows elements of the spaces $\Gamma(\mathcal{G}_\beta)$ and $\Gamma(\mathcal{G}'_\beta)$ will be written as $\boldsymbol{p} = (p_n)$ and $\boldsymbol{u} = (u_n)$, respectively.

For elements of total subset of the space $\Gamma(\mathcal{G}'_\beta)$ let us define the operation $(f^{\otimes n}) \circledast (g^{\otimes n}) := ((f * g)^{\otimes n})$ and extend it onto whole space by linearity and continuity. It is easy to see, that $\Gamma(\mathcal{G}'_\beta)$ is an algebra with respect to $\circledast$. Since the space $\Gamma(\mathcal{G}_\beta)$ is dense in $\Gamma(\mathcal{G}'_\beta)$ (see [2]), the space $\Gamma(\mathcal{G}_\beta)$ also is an algebra with respect to the operation $\circledast$.

[2] O.V. Lopushansky, S.V. Sharyn, Polynomial ultradistributions on cone $\mathbb{R}^d_+$, *Topology*, **48** (2009), no. 2-4, 80 - 90. http://dx.doi.org/10.1016/j.top.2009.11.005

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Using the tensor structure of the space $\Gamma(\mathcal{G}_\beta)$, we extend the Fourier-Laplace transformation onto $\Gamma(\mathcal{G}_\beta)$. First, for elements of total subset of the space $\mathcal{G}_\beta^{\hat{\otimes} n}$ we define the operator $\mathcal{F}^{\otimes n} : \varphi^{\otimes n} \longmapsto \widehat{\varphi}^{\otimes n}$, $\mathcal{F}^{\otimes 0} := I_{\mathbb{C}}$, where $\widehat{\varphi}^{\otimes n} := (F\varphi)^{\otimes n}$. Next, we extend the map $\mathcal{F}^{\otimes n}$ onto whole space $\mathcal{G}_\beta^{\hat{\otimes} n}$ by linearity and continuity. And finally, we define the mapping $\mathcal{F}^{\otimes}$ by the formula

$$\mathcal{F}^{\otimes} := \left(\mathcal{F}^{\otimes n}\right) : \Gamma(\mathcal{G}_\beta) \longrightarrow \Gamma(E_\beta) := \bigoplus_{n \in \mathbb{Z}_+}{}_{fin} E_\beta^{\hat{\otimes} n},$$

$$\boldsymbol{p} = \left(p_n\right) \longmapsto \widehat{\boldsymbol{p}} := \left(\widehat{p}_n\right),$$

where $p_n \in \mathcal{G}_\beta^{\hat{\otimes} n}$, $\widehat{p}_n := \mathcal{F}^{\otimes n} p_n \in E_\beta^{\hat{\otimes} n}$.

Note, that for each $n \in \mathbb{N}$ an element $\widehat{p}_n$ is a symmetric function of n complex variables $\mathbb{C}^n \ni (z_1, \ldots, z_n) \longmapsto \widehat{p}(z_1, \ldots, z_n) \in \mathbb{C}$, i.e. $\widehat{p}_n(z_1, z_2, \ldots, z_n) = \widehat{p}_n(z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(n)})$ for every permutation σ of $\{1, \ldots, n\}$. It implies that elements of the space $\Gamma(E_\beta)$ can be considered as functions $\widehat{\boldsymbol{p}} : \times_{n\in\mathbb{N}} \mathbb{C} \longrightarrow \mathbb{C}$ of infinite many variables

$$\widehat{\boldsymbol{p}}(z_1, \ldots, z_n, \ldots) = \widehat{p}_0 + \sum_{n\in\mathbb{N}} \widehat{p}_n(z_{\mathfrak{b}_n}, \ldots, z_{\mathfrak{e}_n}), \tag{1}$$

where $\mathfrak{b}_n := \frac{n(n-1)}{2} + 1$, $\mathfrak{e}_n := \frac{n(n+1)}{2}$. But we note that actually each function $\widehat{\boldsymbol{p}} \in \Gamma(E_\beta)$ depends on finite (depending on $\widehat{\boldsymbol{p}}$) number of variables, because for each $\widehat{\boldsymbol{p}}$ the sequence in the right hand side of (1) is finite.

Let a countable set of operators $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n, \ldots)$ be defined in a complex Hilbert space $\mathcal{H}$. Note, that we do not assume any commutativity relations. Denote by $\Gamma(\mathcal{H}) := \bigoplus\limits_{n \in \mathbb{Z}_+} \mathcal{H}^{\hat{\otimes} n}$ the symmetric Fock space.

Suppose, that $\mathbf{A}_j$, $j \in \mathbb{N}$, generates an one-parameter (C_0) group $\mathbb{R} \ni t \longmapsto e^{-\mathrm{i}t\mathbf{A}_j} \in \mathscr{L}(\mathcal{H})$, which satisfies the condition

$$\sup_{t \in \mathbb{R}} \|e^{-\mathrm{i}t\mathbf{A}_j}\|_{\mathscr{L}(\mathcal{H})} \leq 1. \tag{2}$$

Let us denote $\mathscr{A}_j := \underbrace{I_{\mathcal{H}} \otimes \cdots \otimes I_{\mathcal{H}}}_{j} \otimes \mathbf{A}_j \otimes I_{\mathcal{H}} \otimes \cdots \in \mathscr{L}(\Gamma(\mathcal{H}))$, $j \in \mathbb{N}$, $A_n := \mathscr{A}_{\mathfrak{b}_n} \otimes \cdots \otimes \mathscr{A}_{\mathfrak{e}_n}$, $n \in \mathbb{N}$. Set $\mathscr{A}_0 := I_{\mathcal{H}} \otimes I_{\mathcal{H}} \otimes \cdots \in \mathscr{L}(\Gamma(\mathcal{H}))$, $A_0 := \mathscr{A}_0$ by definition.

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Instead of the set $\mathbf{A}$ of (noncommuting) operators over Hilbert space $\mathcal{H}$ we consider countable set of commuting operators, acting in the symmetric Fock space $\Gamma(\mathcal{H})$, namely

$$A := (A_0, A_1, A_2, \dots, A_n, \dots). \qquad (3)$$

It easy to see, that each A_n generates strong continuous n-parameter group $\mathbb{R}^n \ni t \longmapsto e^{-\mathrm{i}tA_n} \in \mathscr{L}(\Gamma(\mathcal{H}))$, where $e^{-\mathrm{i}tA_n} := e^{-\mathrm{i}t_1\mathscr{A}_{\mathfrak{b}n}} \otimes \cdots \otimes e^{-\mathrm{i}t_n\mathscr{A}_{\mathfrak{e}n}}$ and $e^{-\mathrm{i}t_i\mathscr{A}_j} := \underbrace{I_{\mathcal{H}} \otimes \ldots \otimes I_{\mathcal{H}}}_{j} \otimes e^{-\mathrm{i}t_i\mathbf{A}_j} \otimes I_{\mathcal{H}} \otimes \ldots$, $i = 1, \dots, n$, $j \in \mathbb{N}$.

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Operator A_n and group $e^{-\mathrm{i}tA_n}$ are defined on whole Fock space $\Gamma(\mathcal{H})$, but they do not act as identity only on $\mathcal{H}^{\hat{\otimes}n}$. So, without restriction of generality we can write $A_n \in \mathscr{L}(\mathcal{H}^{\hat{\otimes}n})$, $e^{-\mathrm{i}tA_n} \in \mathscr{L}(\mathcal{H}^{\hat{\otimes}n})$, $n \in \mathbb{Z}_+$. Let $\mathscr{G}$ be the set of countable systems of operators of view (3). For all $n \in N$ let $\mathscr{G}_n$ be a set of collections of operators of view $A_n = \mathscr{A}_{\mathfrak{b}_n} \otimes \cdots \otimes \mathscr{A}_{\mathfrak{e}_n}$. Set $\mathscr{G}_0 := \{\mathscr{A}_0\}$ by definition.

For all $n \in \mathbb{Z}_+$ let us define the set
$\widetilde{\mathcal{H}}_n := \{\widetilde{p}_n : \mathscr{G}_n \longrightarrow \mathscr{L}(\mathcal{H}^{\hat{\otimes} n}) : p_n \in \mathcal{G}_\beta^{\hat{\otimes} n}\}$, which consist of functions of operator argument

$$\widetilde{p}_n(A_n) := \int_{\mathbb{R}^n} e^{-\mathrm{i} t_1 \mathscr{A}_{\mathfrak{b}_n}} \otimes \cdots \otimes e^{-\mathrm{i} t_n \mathscr{A}_{\mathfrak{e}_n}} p_n(t_1, \ldots, t_n)\, dt_1 \ldots dt_n.$$

Set $\widetilde{p}_0 \colon \mathscr{G}_0 \ni A_0 \longmapsto \widetilde{p}_0(A_0) := p_0 I_{\mathbb{C}} \in \mathscr{L}(\mathbb{C})$ by definition.

Define the map

$$\mathcal{F} := (\mathcal{F}_n) \colon \Gamma(\mathcal{G}_\beta) \ni \boldsymbol{p} = (p_n) \quad \longmapsto \quad \widetilde{\boldsymbol{p}} := \sum_{n \in \mathbb{Z}_+} \widetilde{p}_n \in \widetilde{\mathcal{H}},$$

where $\widetilde{\mathcal{H}} := \sum_{n \in \mathbb{Z}_+} \widetilde{\mathcal{H}}_n$. Condition (2) and [3, Theorem 15.2.1] imply, that all mappings $\mathcal{F}_n : p_n \longmapsto \widetilde{p}_n$, $n \in \mathbb{Z}_+$, are isomorphisms.

[3] E. Hille, R. Phillips, *Functional analysis and semi-groups*, AMS Coll. Publ., vol. XXXI, New York, 1957.

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where $\widetilde{\mathcal{H}} := \sum_{n \in \mathbb{Z}_+} \widetilde{\mathcal{H}}_n$. Condition (2) and [3, Theorem 15.2.1] imply, that all mappings $\mathcal{F}_n : p_n \longmapsto \widetilde{p}_n$, $n \in \mathbb{Z}_+$, are isomorphisms.

[3] E. Hille, R. Phillips, *Functional analysis and semi-groups*, AMS Coll. Publ., vol. XXXI, New York, 1957.

Note, that $\widetilde{\mathcal{H}} := \{\tilde{\boldsymbol{p}} : \mathscr{G} \longrightarrow \mathscr{L}(\Gamma(\mathcal{H})) : \boldsymbol{p} \in \Gamma(\mathcal{G}_\beta)\}$ is an algebra of functions with pointwise multiplication

$$(\tilde{\boldsymbol{p}} \cdot \tilde{\boldsymbol{q}})(A) := \tilde{\boldsymbol{p}}(A) \circ \tilde{\boldsymbol{q}}(A).$$

Theorem

The mapping $\mathcal{F} \colon \Gamma(\mathcal{G}_\beta) \longrightarrow \widetilde{\mathcal{H}}$ acts as homomorphism from the algebra $\{\Gamma(\mathcal{G}_\beta), \circledast\}$ into the algebra $\{\widetilde{\mathcal{H}}, \cdot\}$ of functions of operator argument, defined on the space $\mathscr{G}$.

Results of the article [4] imply that there exists a homomorphism $F^{\otimes} : \Gamma(\mathcal{G}_\beta) \longrightarrow \Gamma(E_\beta)$.

[4] S.V. Sharyn, Paley–Wiener-type theorem for polynomial ultradifferentiable functions, *Carpathian Math. Publ.*, **7** (2015), no. 2, 271 - 279. http://dx.doi.org/10.15330/cmp.7.2.271-279

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Remark

Therefore, the map $\mathcal{F} \circ (F^{\otimes})^{-1} \colon \Gamma(E_\beta) \longrightarrow \widetilde{\mathcal{H}}$ we may treat as "elementary" functional calculus. In other words, we understand the operator $\widetilde{\boldsymbol{p}}(A) = \sum_n \widetilde{p}_n(A_n) \in \mathscr{L}(\Gamma(\mathcal{H}))$ as a "value" of a function $\widehat{\boldsymbol{p}}$ of infinite many variables (see (1)) at a countable system $A = (A_0, A_1, A_2, \dots, A_n, \dots) \in \mathscr{G}$ of operators (see (3)).

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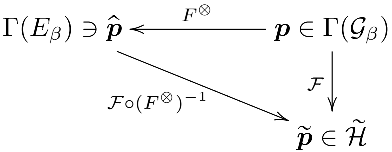

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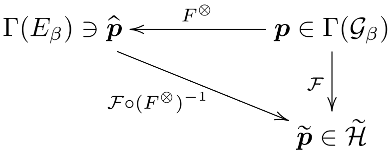

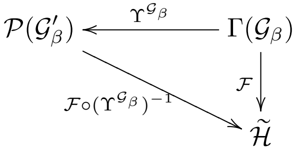

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