A theorem of Helson for general Dirichlet series

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Joint work with Andreas Defant

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02.07.2018

Introduction

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Theorem (Helson, 1968)

Let $f \in H_2(\mathbb{T}^\infty)$ and $\sum a_n n^{-s} := \mathcal{B}(f)$ in view of Bohr's transform. Then for almost all multiplicative characters $\chi\colon \mathbb{N} \to \mathbb{T}$ the Dirichlet series

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$$\overline{\mathbb{R}}, \quad \prod \widehat{(\mathbb{Q}, d)}, \quad \widehat{(U, d)}, \text{ where } U \le \mathbb{R}$$

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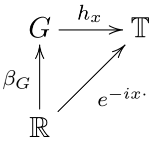

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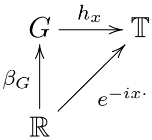

$$\widehat{G} = \{h_x \mid x \in \widehat{\beta_G}(\widehat{G})\}, \quad \text{where} \quad h_x \circ \beta_G = e^{-ix\cdot}$$

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Definition

Let λ be a frequency. Then (G, β_G) is called λ-Dirichlet group if $\lambda \subset \widehat{\beta_G}(\widehat{G})$.

Examples

Two examples:

1) $\mathbb{T}^\infty$ is a Dirichlet group with mapping

$$\beta_{\mathbb{T}^\infty} : \mathbb{R} \to \mathbb{T}^\infty, \quad t \mapsto (p_n^{-it})_n$$

Then

$$\widehat{\beta_{\mathbb{T}^\infty}} : \ \oplus_{n=1}^{\infty} \mathbb{Z} \to \mathbb{R}, \quad \alpha \mapsto \log(p^\alpha)$$

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2)

$$\overline{\mathbb{R}} := \{\gamma \colon \mathbb{R} \to \mathbb{T} \mid \gamma \text{ homomorphism}\},$$

is a Dirichlet group with mapping

$$\beta_{\overline{\mathbb{R}}} \colon \mathbb{R} \to \overline{\mathbb{R}}, \quad x \to [t \mapsto e^{-itx}]$$

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$1 \leq p \leq \infty$:

$$H_p^\lambda(G) := \left\{ f \in L_p(G) \mid \forall x \in \widehat{\beta_G}(\widehat{G})\colon\ \widehat{f}(h_x) \neq 0 \ \Rightarrow\ x \in \lambda \right\}$$

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$$\mathcal{B}\colon H_p^\lambda(G) \hookrightarrow \mathcal{D}(\lambda), \quad f \mapsto \sum \widehat{f}(h_{\lambda_n}) e^{-\lambda_n s}$$

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Theorem (Helson, 1969)

Let $f \in H_2^\lambda(\overline{\mathbb{R}})$*,* λ *satisfies* (BC) *and* $\sum a_n e^{-\lambda_n s} := \mathcal{B}(f)$ *in view of Bohr's transform. Then for almost all* $\omega \in \overline{\mathbb{R}}$ *the Dirichlet series*

$$\sum a_n h_{\lambda_n}(\omega) e^{-\lambda_n s}$$

converges on $[Re > 0]$*.*

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Bohr's condition (BC):

$$\exists\, C, l \;\forall n \in \mathbb{N}: \quad \lambda_{n+1} - \lambda_n \geq C e^{-\lambda_n l}$$

A maximal inequality - Helson's theorem in $H_p^\lambda(G)$

Theorem

Let $f \in H_p^\lambda(G)$, $1 < p < \infty$, λ *satisfies* (BC) *and* $\sum a_n e^{-\lambda_n s} := \mathcal{B}(f)$ *in view of Bohr's transform. Then for all* $u > 0$ *there is a constant* $C = C(u, \lambda, p)$ *such that*

$$\left(\int_G \sup_N \left| \sum_{n=1}^N a_n h_{\lambda_n}(\omega) e^{-u\lambda_n} \right|^p dm(\omega) \right)^{\frac{1}{p}} \leq C \|f\|_p.$$

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Corollary

For all λ*-groups* (G, β_G) *for almost all* $\omega \in G$ *the Dirichlet series*

$$D^\omega(s) = \sum a_n h_{\lambda_n}(\omega) e^{-\lambda_n s}$$

converges on $[Re > 0]$*.*

A maximal inequality - Helson's theorem in $H_p^\lambda(G)$

Corollary ($\lambda_n = \log(n)$)

Let $1 \leq p \leq \infty$, $f \in H_p(\mathbb{T}^\infty)$ and $\sum a_n n^{-s} = \mathcal{B}(f)$ in view of Bohr's transform. Then for all $u > 0$ there is a constant $C > 0$ such that

$$\left(\int_\Omega \sup_N \left| \sum_{n=1}^N a_n \chi(n) n^{-u} \right|^p d\chi \right)^{\frac{1}{p}} \leq C \|f\|_p.$$

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Corollary ($\lambda_n = \log(n)$)

$$\left(\int_{\mathbb{T}^\infty} \sup_N \left| \sum_{1 \leq p^\alpha \leq N} \hat{f}(\alpha)(p^{-u} z)^\alpha \right|^p dz \right)^{\frac{1}{p}} \leq C \|f\|_p.$$