

# Spectra of weighted composition operators on spaces of analytic functions

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## Coauthors and references

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*Spectra of weighted composition operators on abstract Hardy spaces*

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# Composition operators

Let  $\varphi \in H(\mathbb{D})$  be a self-map of  $\mathbb{D}$ , i.e.,  $\varphi(\mathbb{D}) \subset \mathbb{D}$ . The map

$$C_\varphi: H(\mathbb{D}) \rightarrow H(\mathbb{D})$$

given by

$$C_\varphi f(z) = f \circ \varphi(z), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}$$

is called a **composition operator**.

## Composition operators – properties

- **Boundedness** (due to Littlewood, '20 of XX century) – every self-map of  $\mathbb{D}$  generates a bounded composition operator  $C_\varphi: H^p \rightarrow H^p$  (irrespective of  $p$ ,  $p \in [1, \infty]$ ).
- **Compactness** (due to J. Shapiro, '80 of XX century) – a composition operator  $C_\varphi$  is compact on  $H^p$  (irrespective of  $p$ ,  $p \in [1, \infty]$ ) if and only if

$$\lim_{|z| \rightarrow 1^-} \frac{N_\varphi(z)}{\log \frac{1}{|z|}} = 0,$$

where  $N_\varphi$  is a Nevanlinna counting function.

# Weighted composition operators

Let  $\varphi \in H(\mathbb{D})$  be a self-map of  $\mathbb{D}$  and  $u \in H(\mathbb{D})$ . Then the map

$$uC_\varphi: H(\mathbb{D}) \rightarrow H(\mathbb{D})$$

is called a **weighted composition operator**.

A map  $uC_\varphi$  is a bounded operator on  $H^p$  if and only if  $u$  is a bounded function.

## Research task

## Challenge

*Study the spectral properties of weighted composition operators on spaces of holomorphic functions (e.g., Hardy, Bloch spaces etc).*

## Notation

- For a self-map of  $\mathbb{D}$  the symbol  $\varphi_n$  denotes the  $n$ -th iterate of  $\varphi$ , i.e.,

$$\varphi_n = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n \text{ times}}$$

- For any  $n$

$$(uC_\varphi)^n f(z) = u(z)u(\varphi(z)) \dots u(\varphi_{n-1}(z))f(\varphi_n(z)) \quad f \in H(\mathbb{D}), z \in \mathbb{D}$$

and

$$(uC_\varphi)^n = u_{(n)}C_{\varphi_n},$$

where

$$u_{(n)} = \prod_{j=0}^{n-1} u \circ \varphi_j \in H(\mathbb{D}), \quad n \in \mathbb{N}.$$

## Köthe function spaces

$X$  is a **Köthe function spaces** if

- $X \subset L^0(\Omega, \Sigma, \mu)$  – the space of real valued measurable functions on  $\Omega$ .  
The order  $|x| \leq |y|$  means  $|x(\omega)| \leq |y(\omega)|$  for  $\mu$ -almost all  $\omega \in \Omega$ .
- There exists  $u \in X$  with  $u > 0$   $\mu$ -a.e. on  $\Omega$  and  $|x| \leq |y|$  with  $x \in L^0(\Omega)$  and  $y \in X$  implies  $x \in X$  with  $\|x\|_X \leq \|y\|_X$ .
- If  $f \in X$ , then for any  $g$  equimeasurable with  $f$ ,  $\|f\|_X = \|g\|_X$

We will consider **complex** Köthe function spaces.

The role model for  $X$  are Lebesgue spaces, other important examples

- Orlicz spaces
- Lorentz spaces
- Marcinkiewicz spaces

Köthe function space  $X$  is **maximal** (or has the **Fatou property**) if whenever  $\{x_n\}$  is a norm bounded sequence in  $X$  such that  $0 \leq x_n \uparrow x \in L^0(\Omega)$ , then  $x \in X$  and  $\|x\| = \lim_{n \rightarrow \infty} \|x_n\|$ .

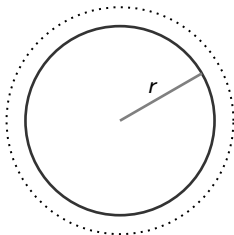


## Abstract Hardy spaces

Let  $X$  be the Köthe space on  $\mathbb{T} := [0, 2\pi)$ . We define **abstract Hardy spaces**  $HX(\mathbb{D})$  in the following way.

$$HX(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_{HX(\mathbb{D})} := \sup_{r \in [0,1)} \|f_r\|_X < \infty \right\}$$

where  $f_r(t) := f(re^{it})$ ,  $t \in \mathbb{T}$ .



## Multipliers of abstract Hardy spaces

For any function  $u \in H(\mathbb{D})$ , an operator

$$M_u: H(\mathbb{D}) \rightarrow H(\mathbb{D}), \quad M_u f(z) = u(z)f(z), \quad f \in H(\mathbb{D})$$

is called a **multiplication operator**.

## Theorem

Let  $X$  be a maximal r.i. space on  $\mathbb{T}$  and suppose that  $u \in H(\mathbb{D})$ . Then the following statements are equivalent:

- $uHX \subset HX$
- $u \in HX$  and the operator  $M_u$  is bounded
- $u \in H^\infty$ .

Moreover,  $\|M_u: HX \rightarrow HX\| = \|u\|_\infty$ .

## Automorphisms of the discs

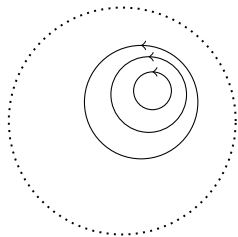
$\varphi$  – nontrivial (i.e., not identity) **automorphism** of  $\mathbb{D}$

- **elliptic** –  $\varphi$  has a unique fixed point in  $\mathbb{D}$ ,
- **parabolic** –  $\varphi$  has a unique fixed (Denjoy–Wolf) point  $a \in \partial\mathbb{D}$ ,  $\varphi'(a) = 1$ ,
- **hyperbolic** –  $\varphi$  has two distinct fixed points in  $\partial\mathbb{D}$ , an attractive fixed (Denjoy–Wolf) point  $a$  and a repulsive fixed point  $b$ ,  $\varphi'(a) \in (0, 1)$  and  $\varphi'(b) = 1/\varphi'(a)$ .

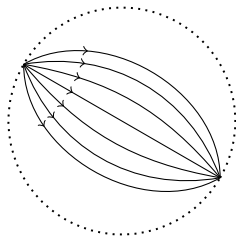
When  $\varphi$  is a parabolic or hyperbolic automorphism, then

$$\lim_{n \rightarrow \infty} (1 - |\varphi_n(0)|)^{\frac{1}{n}} = \varphi'(a).$$

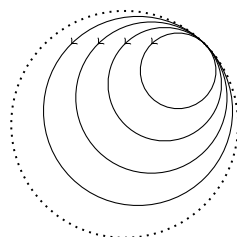
## Automorphisms of the disc – pictures



elliptic



hyperbolic



parabolic

## Abstract Hardy spaces – elliptic case

## Theorem

Suppose that  $u \in A(\mathbb{D})$  and  $\varphi$  is an elliptic automorphism with the unique fixed point  $a \in \mathbb{D}$ .

- If there is a positive integer  $j$  such that  $\varphi_j(z) = z$  for all  $z \in \mathbb{D}$ , then letting  $m$  be the smallest such integer, we have

$$\sigma_{HX}(uC_\varphi) = \overline{\{\lambda \in \mathbb{C} : u_{(m)}(z) = \lambda^m \text{ for some } z \in \mathbb{D}\}}.$$

- If  $\varphi_n \neq \text{Id}$  for every  $n \in \mathbb{N}$  and if  $uC_\varphi : HX \rightarrow HX$  is invertible, then

$$\sigma_{HX}(uC_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| = |u(a)|\}.$$

## Abstract Hardy spaces – parabolic case

## Theorem

Let  $X$  be a maximal r.i. space on  $\mathbb{T}$  and  $uC_\varphi: HX \rightarrow HX$  be an invertible composition operator and assume that  $\varphi$  is a **parabolic automorphism** of the disc  $\mathbb{D}$  with a unique fixed point  $a \in \partial\mathbb{D}$  and  $u \in A(\mathbb{D})$ . Then

$$r_{HX}(uC_\varphi) = |u(a)|$$

and

$$\sigma_{HX}(uC_\varphi) = \{z \in \mathbb{C} : |z| = |u(a)|\}.$$

## Invertible weighted composition operators

## Theorem

Let  $X$  be an r.i. space on  $\mathbb{T}$  and suppose that  $uC_\varphi: HX \rightarrow HX$  is a weighted composition operator. Then  $uC_\varphi$  is **invertible** if and only if  $u \in H^\infty$ ,  $u$  is bounded away from zero on  $\mathbb{D}$ , and  $\varphi$  is an automorphism of  $\mathbb{D}$ . Moreover, if this is the case, then the inverse is also a weighted composition operator and

$$(uC_\varphi)^{-1} = \frac{1}{u \circ \varphi^{-1}} C_{\varphi^{-1}}.$$



P. Bourdon

*Invertible weighted composition operators*  
Proc. Amer. Math. Soc. 142 (2013), 289–299.

## Hardy–Lorentz spaces – stability of the spectrum

Given a non-atomic measure space  $(\Omega, \Sigma, \mu)$  and  $p \in (1, \infty)$ ,  $q \in [1, \infty]$ , the **Lorentz space**  $L^{p,q}$  on  $(\Omega, \Sigma, \mu)$  consists of all  $L^0(\Omega)$  such that

$$\|f\|_{L^{p,q}} = \begin{cases} \left( \int_0^{\mu(\Omega)} (t^{1/p} f^{**}(t))^q \frac{dt}{t} \right)^{1/q}, & q < \infty \\ \sup_{t \in (0, \mu(\Omega))} t^{1/p} f^{**}(t), & q = \infty, \end{cases}$$

where  $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$ ,  $t > 0$  and  $f^*$  denotes the non-increasing rearrangement of  $f$  (i.e.,  $f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}$ ,  $t \geq 0$ ).

Recall that for  $p \in (1, \infty)$ ,  $q \in [1, \infty]$ , the **Hardy–Lorentz space**  $H^{p,q}$  is given by  $H^{p,q} = HL^{p,q}$ .

## Theorem

Let  $\varphi$  be a holomorphic self-map of  $\mathbb{D}$ ,  $u \in H^\infty$ , and  $p, q \in (1, \infty)$ . Then

$$\sigma_{H^{p,q}}(uC_\varphi) = \sigma_{H^p}(uC_\varphi).$$



## Hardy–Lorentz spaces – automorphisms

## Corollary

Let  $uC_\varphi$  be an invertible operator on Hardy–Lorentz space  $H^{p,q}$ ,  $p, q \in (1, \infty)$  and  $u \in A(\mathbb{D})$ . Then

- if  $\varphi$  is a parabolic automorphism with fixed point  $s \in \partial\mathbb{D}$ , then

$$\sigma_{H^{p,q}}(uC_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| = |u(a)|\},$$

- if  $\varphi$  is a hyperbolic automorphism with attractive point  $a \in \partial\mathbb{D}$  and repulsive point  $b \in \partial\mathbb{D}$ , then

$$\sigma_{H^{p,q}}(uC_\varphi) = \left\{ \lambda \in \mathbb{C} : \min \left\{ \frac{|u(a)|}{\varphi'(a)^{1/p}}, \frac{|u(b)|}{\varphi'(b)^{1/p}} \right\} \leq |\lambda| \leq \max \left\{ \frac{|u(a)|}{\varphi'(a)^{1/p}}, \frac{|u(b)|}{\varphi'(b)^{1/p}} \right\} \right\}.$$



O. Hyvärinen, I. Nieminen

*Essential spectra of weighted composition operators with hyperbolic symbols*  
Concr. Oper. 2 (2015), 110–119.

## Hardy–Lorentz spaces – non-automorphisms

## Corollary

Assume that  $u, \varphi \in H(\mathbb{D})$ ,  $\varphi(\mathbb{D}) \subset \mathbb{D}$ , and  $a$  is a Denjoy–Wolff point with  $\varphi_n \rightarrow a$  uniformly in  $\mathbb{D}$  as  $n \rightarrow \infty$  and  $\varphi'(a) < 1$ . If  $u$  is bounded in  $\mathbb{D}$  and continuous at  $a$ , then

$$\sigma_{HP,q}(uC_\varphi) = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{|u(a)|}{|\varphi'(a)|^{1/p}} \right\}.$$



O. Hyvärinen, I. Nieminen

*Essential spectra of weighted composition operators with hyperbolic symbols*  
Concr. Oper. 2 (2015), 110–119.



C. Cowen, E. Ko, D. Thompson, F. Tian

*Spectra of some weighted composition operators on  $H^2$*   
Acta Sci. Math. (Szeged) 82 (2016), no. 1–2, 221–234.

## Proofs ingredients – interpolation sequences and interpolation

- **Complex functions theory.** A sequence  $\{z_j\} \subset \mathbb{D}$  is called interpolating for  $H^\infty(\mathbb{D})$  if for any bounded sequence of complex numbers  $\{c_j\}$ , there is a function  $f \in H^\infty(\mathbb{D})$  such that  $f(z_j) = c_j$ . It is known that if  $z_0 \in \mathbb{D}$ , then the sequence of iterates  $\{\varphi_n(z_0)\}$ ,  $n \in \mathbb{N}$ , is an interpolating sequence for  $H^\infty(\mathbb{D})$ .
- **Function spaces methods.** In particular fundamental function of Köthe function space  $X$

$$t \mapsto \phi(t) = \|\chi_{[0,t]}\|, \quad t \in \{\mu(A) : A \in \Sigma\}.$$

- **Interpolation of operators.** For all  $p_0, p_1 \in (1, \infty)$ ,  $p_0 \neq p_1$ ,  $q_0, q_1, r \in [1, \infty]$ ,  $s \in (0, 1)$ ,

$$(H^{p_0, q_0}, H^{p_1, q_1})_{s, r} = H^{p_s, r},$$

where  $1/p_s = (1-s)/p_0 + s/p_1$ .

## Proofs ingredients – interpolation sequences and interpolation

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