

ERGODIC PROPERTIES OF THE MULTIPLICATION OPERATOR

Alberto Rodríguez Arenas

Work supervised by José Bonet and Enrique Jordá
IUMPA – Universitat Politècnica de València

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Given $T \in \mathcal{L}(E)$, the averages are $T_{[n]} := \frac{1}{n} \sum_{m=1}^n T^m$, $n \in \mathbb{N}$.
The operator T is said to be **(uniformly) mean ergodic** if $\{T_{[n]}\}_{n=1}^{\infty}$ converges in $\mathcal{L}_s(E)$ (resp. $\mathcal{L}_b(E)$).

Weighted spaces of continuous functions

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Klilou, Manhas, Oubbi, Singh 2000's

For holomorphic functions on \mathbb{D} : Bonet and Ricker 2009

Power Boundedness

$$T_\varphi \text{ is power bounded} \iff S_\varphi \text{ is power bounded} \iff \|\varphi\|_\infty \leq 1$$

Mean Ergodicity

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$$((M_\varphi)_{[n]}f)(x) = \begin{cases} \frac{f(x)\varphi(x)}{n} \frac{1-\varphi(x)^n}{1-\varphi(x)}, & \text{if } \varphi(x) \neq 1 \\ f(x), & \text{if } \varphi(x) = 1 \end{cases}$$

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$$h_f(x) = \begin{cases} 0, & \text{if } \varphi(x) \neq 1 \\ f(x), & \text{if } \varphi(x) = 1 \end{cases}$$

Mean Ergodicity in C_V and C_V^0

Theorem

$S_\varphi \in \mathcal{L}(C_V^0)$ is mean ergodic $\iff \|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.

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Theorem

The following assertions are equivalent:

- 1 $T_\varphi \in \mathcal{L}(C_V)$ is mean ergodic,
- 2 $T_\varphi \in \mathcal{L}(C_V)$ is uniformly mean ergodic,
- 3 $S_\varphi \in \mathcal{L}(C_V^0)$ is uniformly mean ergodic,
- 4 $\|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ is open and

$$\inf\{|1 - \varphi(x)| : x \in X \setminus \varphi^{-1}(1)\} > 0.$$

Inductive Limits

$V = (v_k)_k$ is a decreasing family of continuous weights.

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Nachbin Family

$\bar{V} := \{ \bar{v} : X \rightarrow (0, \infty) : \bar{v} \text{ is upper semicontinuous and } \frac{\bar{v}}{v_k} \text{ is bounded on } X \forall k \in \mathbb{N} \}$.

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$$C\bar{V} := \{f \in C(X) : p_{\bar{v}}(f) := \sup_{x \in X} \bar{v}(x)|f(x)| < \infty, \forall \bar{v} \in \bar{V}\}$$

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Bierstedt, Bonet, Meise, Summers 1980's

Continuity

$$M_\varphi \in \mathcal{L}(VC) \iff M_\varphi \in \mathcal{L}(V_0C)$$
$$\iff \forall k \exists l : \sup_{x \in X} \frac{v_l(x)}{v_k(x)} |\varphi(x)| < \infty$$

Power Boundedness

$$\begin{aligned} M_\varphi \in \mathcal{L}(VC) \text{ power bounded} &\iff M_\varphi \in \mathcal{L}(V_0C) \text{ power bounded} \\ &\iff \|\varphi\|_\infty \leq 1 \end{aligned}$$

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$M_\varphi \in \mathcal{L}(V_0 C)$ is mean ergodic $\iff \|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.

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$M_\varphi \in \mathcal{L}(VC)$ is mean ergodic $\iff M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic.

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If V has property (S), then: $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic $\iff \|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.

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Theorem

If V has property (S), then: $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic $\iff \|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.

(S) $\iff VC = V_0C$

Mean Ergodicity using \overline{V}

Theorem

$M_\varphi \in \mathcal{L}(V_0 C)$ is uniformly mean ergodic $\iff \|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ is open in X and for every $k \in \mathbb{N}$ and every $\bar{v} \in \overline{V}$,

$$\lim_{n \rightarrow \infty} \sup_{x \in Y} \frac{1}{n} \frac{\bar{v}(x)}{v_k(x)} |\varphi(x)| \frac{|1 - \varphi(x)^n|}{|1 - \varphi(x)|} = 0,$$

where $Y = X \setminus \varphi^{-1}(1)$.

Mean Ergodicity using (D)

Theorem

If V has property (D), then: $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic $\iff \|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ is open in X and for every $k \in \mathbb{N}$ and every $\bar{v} \in \bar{V}$,

$$\lim_{n \rightarrow \infty} \sup_{x \in Y} \frac{1}{n} \frac{\bar{v}(x)}{v_k(x)} |\varphi(x)| \frac{|1 - \varphi(x)^n|}{|1 - \varphi(x)|} = 0,$$

where $Y = X \setminus \varphi^{-1}(1)$.

$$(D) \iff VC = C\bar{V}$$

Projective Limits

$A = (a_k)_k$ is an increasing family of continuous weights.

$$CA = \text{proj}_{n \in \mathbb{N}} C_{a_n}, \quad CA_0 = \text{proj}_{n \in \mathbb{N}} C_{a_n}^0$$

Continuity

$$M_\varphi \in \mathcal{L}(CA) \iff M_\varphi \in \mathcal{L}(CA_0)$$
$$\iff \forall l \exists k : \sup_{x \in X} \frac{a_l(x)}{a_k(x)} |\varphi(x)| < \infty$$

Power Boundedness

$$M_\varphi \in \mathcal{L}(CA) \text{ power bounded} \iff M_\varphi \in \mathcal{L}(CA_0) \text{ power bounded} \\ \iff \|\varphi\|_\infty \leq 1$$

Mean Ergodicity

Theorem

$M_\varphi \in \mathcal{L}(CA_0)$ is mean ergodic $\iff \|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.

Theorem

If X is even σ -compact, then the following assertions are equivalent:

- 1 $M_\varphi \in \mathcal{L}(CA)$ is mean ergodic,
- 2 $M_\varphi \in \mathcal{L}(CA)$ is uniformly mean ergodic,
- 3 $M_\varphi \in \mathcal{L}(CA_0)$ is uniformly mean ergodic,
- 4 $\|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ is open in X and for every $k \in \mathbb{N}$ and every $\bar{v} \in \bar{V}$,

$$\lim_{n \rightarrow \infty} \sup_{x \in Y} \frac{\bar{v}(x) a_k(x)}{n} |\varphi(x)| \frac{|1 - \varphi(x)^n|}{|1 - \varphi(x)|} = 0,$$

where $Y = X \setminus \varphi^{-1}(1)$.

Spectrum

E l.c.H.s.

The **resolvent set** $\rho(T, X)$ of $T \in \mathcal{L}(X)$ consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of T is the set $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$.

The **point spectrum** is the set $\sigma_{pt}(T, X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective.

Spectrum

Proposition

Let E be either C_v , C_v^0 , VC , V_0C , CA or CA_0 , for a weight v .
Then

$$\sigma_{pt}(M_\varphi, E) \subset \{\varphi(x) : x \in X\} \subset \sigma(M_\varphi, E).$$

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Let E be either C_v , C_v^0 , VC , V_0C , CA or CA_0 , for a weight v .
Then

$$\sigma_{pt}(M_\varphi, E) \subset \{\varphi(x) : x \in X\} \subset \sigma(M_\varphi, E).$$

Proposition

Let E be either C_v or C_v^0 , for a weight v . Then

$$\sigma(M_\varphi, E) = \overline{\{\varphi(x) : x \in X\}}.$$

Spectrum

Proposition

Let E be either VC, V_0C , CA or CA_0 . Then the following assertions are equivalent:

- (1) $\mu \in \rho(M_\varphi, E)$,
- (2) $M_\phi \in \mathcal{L}(X)$, with $\phi(\cdot) = \frac{1}{\varphi(\cdot) - \mu}$,
- (3) for each k , there exists $l \geq k$ such that

$$\sup_{x \in X} \frac{v_l(x)}{v_k(x)} \cdot \frac{1}{|\varphi(x) - \mu|} = \sup_{x \in X} \frac{a_k(x)}{a_l(x)} \cdot \frac{1}{|\varphi(x) - \mu|} < \infty.$$

*-spectrum of Waelbroeck

E l.c.H.s.

$\rho^*(T, E)$ consists of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that each $\mu \in B(\lambda, \delta)$ belongs as well to $\rho(T, E)$ and such that the set $\{(\mu I - T)^{-1} : \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(E)$.

$$\sigma^*(T, E) := \mathbb{C} \setminus \rho^*(T, E).$$

Theorem

Let E be either VC , V_0C , CA or CA_0 . Then

$$\sigma^*(M_\varphi, E) = \overline{\sigma(M_\varphi, E)}$$

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