Convolution operators in discrete Cesàro spaces

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The Cesàro operator C

Consider the *Cesàro operator* $C : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ given by

$$C(x) := \left(x_0, \frac{x_0 + x_1}{2}, \frac{x_0 + x_1 + x_2}{3}, \ldots\right), \qquad x = (x_n)_0^\infty \in \mathbb{C}^{\mathbb{N}}.$$

- $|C(x)| \leq C(|x|), \qquad \forall x \in \mathbb{C}^{\mathbb{N}}$ (with $|x| := (|x_n|)_0^\infty$)
- C is a vector space isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto $\mathbb{C}^{\mathbb{N}}$.

The discrete Cesàro space for $1 < p < \infty$ (early 1970's):

$$\mathrm{ces}(p) := \left\{ x \in \mathbb{C}^{\mathbb{N}} : \|x\|_{\mathrm{ces}(p)} := \left\| \left(\frac{1}{n+1} \sum_{k=0}^{n} |x_k| \right)_0^\infty \right\|_p = \|C(|x|)\|_p < \infty \right\}$$

Intense research by G. Bennett, Mem. Amer. Math. Soc. 120 (1996): "Factorizing the classical inequalities"

Properties of $\mathrm{ces}(p)$, $1 < p < \infty$

- $(\mathrm{ces}(p), \|\cdot\|_{\mathrm{ces}(p)})$ is a **reflexive** Banach lattice.
  $e_k := (\delta_{nk})_{n=0}^{\infty}$, $k \in \mathbb{N}$ are an **unconditional basis**:

$$\|e_k\|_{\mathrm{ces}(p)} \simeq (k+1)^{-1/p'}, \quad k \in \mathbb{N}, \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

- Equivalent norms in $\mathrm{ces}(p)$:
  (a) $x \mapsto \left\| \left(\sum_{k=n}^{\infty} \frac{|x_k|}{k+1} \right)_{n=0}^{\infty} \right\|_p$.
  (b) $x \mapsto \left(|x_0|^p + \sum_{j=0}^{\infty} 2^{j(1-p)} \left[\sum_{k=2^j}^{2^{j+1}-1} |x_k| \right]^p \right)^{1/p}$.
- Hardy's inequality for $1 < p < \infty$: $\|C(|x|)\|_p \le p'\|x\|_p$, $x \in \ell^p$.
  $\Rightarrow C$ maps $\ell^p \to \ell^p$ continuously (operator norm is p').

Moreover, the inclusion $\ell^p \subseteq \mathrm{ces}(p)$ is *proper*, continuous and C maps $\mathrm{ces}(p) \to \ell^p$ continuously (isometrically).
Hence, also C maps $\mathrm{ces}(p) \to \mathrm{ces}(p)$ continuously.

Further properties of $\mathrm{ces}(p), 1 < p < \infty$

- Let $1 < p < \infty$ and $x \in \mathbb{C}^{\mathbb{N}}$. Remarkable property (G. Bennett):
  $$\mathbf{x} \in \mathrm{ces}(p) \ \textbf{ if and only if } \mathbf{C}(|\mathbf{x}|) \in \mathrm{ces}(p)$$
  $\ell^p, 1 < p < \infty$, do **not** have this property.
- G. Curbera (2014): The **largest** of all those **solid** Banach lattices $X \subseteq \mathbb{C}^{\mathbb{N}}$ with $\ell^p \subseteq X$ s.t. C maps $X \to \ell^p$ cont. is $\mathrm{ces}(p)$.
- G. Curbera (2014): Largest amongst the class of spaces ℓ^r $(1 < r < \infty)$ satisfying $\ell^r \subseteq \mathrm{ces}(p)$ is the space ℓ^p.
- Dual Banach space $\mathrm{ces}(p)^*$ identified by A.A. Jagers (1974). Rather complicated: G. Bennett (1996) showed:
  $$\mathrm{ces}(p)^* \simeq d(p') = \left\{x \in \mathbb{C}^{\mathbb{N}} : \textstyle\sum_{n=0}^{\infty} \sup_{k \geq n} |x_k|^{p'} < \infty\right\}$$
  for the equivalent (but **not** dual) norm
  $$\|x\|_{d(p')} := \left\|(\sup_{k \geq n} |x_k|)_{n=0}^{\infty}\right\|_{p'} := \left\|\hat{x}\right\|_{p'}.$$

Convolution operators in ces(p)

Fix $b = (b_n)_{n=0}^{\infty} \in \mathbb{C}^{\mathbb{N}}$. Each element

$$x * b := \left(\sum_{j=0}^{n} x_j b_{n-j} \right)_{n=0}^{\infty}, \qquad x \in \mathbb{C}^{\mathbb{N}},$$

again belongs to $\mathbb{C}^{\mathbb{N}}$. So, we have the **convolution operator**

$$T_b : x \mapsto x * b = b * x,$$

which is well defined and linear from $\mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$. Moreover,

$$T_b T_c = T_c T_b = T_{b*c}, \qquad b, c \in \mathbb{C}^{\mathbb{N}}.$$

Relevant are the following identities:

- $T_{e_0} = I$ (Identity operator), i.e. $x * e_0 = x, \ \forall x \in \mathbb{C}^{\mathbb{N}}$.
- $e_n = e_1 * e_1 * \ldots * e_1 \quad (n \text{ terms}), \quad \forall n \geq 1.$

We first recall when T_b acts in the spaces ℓ^p, $1 < p < \infty$.

Convolution p-multipliers for $\ell^p, 1 \le p \le \infty$

$b \in \mathbb{C}^{\mathbb{N}}$ is a **(convolution)** ***p*-multiplier** for ℓ^p (write $b \in \mathcal{M}(\ell^p)$) if

$$x * b \in \ell^p, \qquad \forall x \in \ell^p.$$

Closed graph theorem $\Rightarrow T_b : \ell^p \to \ell^p$ is continuous.
Facts: [N.K. Nikolskii (1966) & ($p = 2$) I. Schur (1917)]

- $\mathcal{M}(\ell^1) = \mathcal{M}(\ell^\infty) = \ell^1$.
- $\mathcal{M}(\ell^p) = \mathcal{M}(\ell^{p'})$
  If $1 \le p_1 < p_2 \le 2$, then $\mathcal{M}(\ell^{p_1}) \subsetneqq \mathcal{M}(\ell^{p_2})$.
- $\ell^1 \subsetneqq \mathcal{M}(\ell^p) \subsetneqq \widehat{H}^\infty$, whenever $1 < p < \infty$ (and $\mathcal{M}(\ell^p) \subsetneqq \ell^p$).
- **Schur:** $\mathcal{M}(\ell^2) = \widehat{H}^\infty$, i.e., each element of $\mathcal{M}(\ell^2)$ is the sequence of Taylor coefficients of some function from $H^\infty(\mathbb{D})$.

What if we replace $\ell^p, 1 < p < \infty$, with $\mathrm{ces}(p), 1 < p < \infty$?

Convolution p-multipliers for ces(p), $1 < p < \infty$

Which elements $b \in \mathbb{C}^{\mathbb{N}}$ satisfy (for $1 < p < \infty$ fixed)

$$x * b \in \mathrm{ces}(p), \qquad \forall x \in \mathrm{ces}(p)?$$

Equivalently, when does T_b map $\mathrm{ces}(p) \to \mathrm{ces}(p)$ continuously?
Answer is dramatically different than for ℓ^p spaces.

Proposition 1 [Curbera (2014)]

Let $1 < p < \infty$ and $b \in \mathbb{C}^{\mathbb{N}}$. Then $T_b : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ maps $\mathrm{ces}(p)$ into $\mathrm{ces}(p)$, i.e. $b \in \mathscr{M}(\mathrm{ces}(p))$, if and only if $b \in \ell^1$. In this case

$$\|T_b\|_{\mathrm{op}} = \|b\|_{\ell^1} = \sum_{n=0}^{\infty} |b_n|\,.$$

$\mathscr{L}_p := (L(\mathrm{ces}(p)), \|\cdot\|_{\mathrm{op}})$ is a unital, **non-commutative** Banach algebra (for composition of operators from $\mathrm{ces}(p) \to \mathrm{ces}(p)$).

How does one identify the **spectrum** $\sigma(T_b)$ of $T_b \in \mathscr{L}_p$?

Proposition 1 implies $\mathscr{M}_p := \left\{T_b : b \in \mathscr{M}(\mathrm{ces}(p)) = \ell^1\right\}$ is a unital, commutative, closed (proper) **subalgebra** of $(\mathscr{L}_p, \|\cdot\|_{\mathrm{op}})$.
$\mathscr{M}_p$ is isometrically isomorphic to the B-algebra $\mathscr{A} := (\ell^1, *)$.
Here, $*$ is convolution: the unit is $e_0 = (1, 0, 0, \ldots)$.

$\mathscr{R} \subseteq C(\overline{\mathbb{D}})$ is the unital, commutative B-algebra of functions

$$\varphi_b(z) := \sum_{n=0}^{\infty} b_n z^n, \qquad z \in \overline{\mathbb{D}},$$

for all $b \in \ell^1$. We use pointwise operations of scalar multiplication, addition and product: the norm is

$$\|\varphi_b\|_{\mathscr{R}} := \|b\|_{\ell^1} .$$

Unit of $\mathscr{R}$ is the constant function **1**. As $\varphi_b \varphi_c = \varphi_{b*c}$, the map $b \mapsto \varphi_b$ is an isometric B-algebra isomorphism of $\mathscr{A}$ onto $\mathscr{R}$.

For a B-algebra $\mathscr{B}$ with unit e and $u \in \mathscr{B}$, define

$$\sigma_{\mathscr{B}}(u) := \{\lambda \in \mathbb{C} : u - \lambda e \text{ not invertible}\} \text{ and } \rho_{\mathscr{B}}(u) := \mathbb{C} \setminus \sigma_{\mathscr{B}}(u).$$

The B-algebras $\mathscr{M}_p \simeq \mathscr{A} \simeq \mathscr{R}$ $(1 < p < \infty)$ are all isometrically isomorphic. So the **spectrum** of T_b satisfies

$$\sigma_{\mathscr{M}_p}(T_b) = \sigma_{\mathscr{A}}(b) = \sigma_{\mathscr{R}}(\varphi_b), \qquad \forall b \in \ell^1. \tag{$\star$}$$

Maximal ideal space of $\mathscr{R}$ is $\overline{\mathbb{D}}$. Gelfand theory and $(\star)$ imply:

$$\sigma_{\mathscr{M}_p}(T_b) = \varphi_b(\overline{\mathbb{D}}) = \{\varphi_b(z) : |z| \le 1\}, \qquad b \in \ell^1 \ (1 < p < \infty).$$

$\mathscr{M}_p$ is a closed, unital **subalgebra** of the non-commutative unital B-algebra $\mathscr{L}_p$. Consequently,

$$\sigma_{\mathscr{L}_p}(T_b) \subseteq \sigma_{\mathscr{M}_p}(T_b), \qquad b \in \ell^1.$$

If the more traditional notation $\sigma(T_b)$ is used for

$$\sigma_{\mathscr{L}_p}(T_b) = \left\{\lambda \in \mathbb{C} : (T_b - \lambda I) \text{ not invertible in } \mathscr{L}_p\right\},$$

the previous containment becomes

$$\sigma(T_b) \subseteq \{\varphi_b(z) : |z| \le 1\}, \qquad b \in \ell^1.$$

How does one deduce this is an equality?

The shift-operator in ces(p), $1 < p < \infty$

The **right-shift** operator S_p in ces(p), given by

$$S_p\left((x_n)_{n=0}^{\infty}\right) := (0, x_0, x_1, \ldots), \qquad x \in \mathrm{ces}(p),$$

satisfies $\left\|S_p\right\|_{\mathrm{op}} = 1$ and the identity (with $e_1 = (0, 1, 0, 0, \ldots)$)

$$S_p\left(x\right) = T_{e_1}(x) = x * e_1, \qquad x \in \mathrm{ces}(p)\,.$$

This formula, in turn, implies that

$$S_p^n = T_{e_n} = T_{e_1 * \ldots e_1}, \qquad (n \text{ times convolution.})$$

Via the isomorphism $\mathrm{ces}(p)^* \simeq d(p')$, the **adjoint operator** $S_p^* : \mathrm{ces}(p)^* \to \mathrm{ces}(p)^*$ can be identified with the **left-shift** operator $L_{p'}$ in $d(p')$ given by

$$u \mapsto L_{p'}\left((u_n)_{n=0}^{\infty}\right) := (u_1, u_2, u_3, \ldots), \qquad u \in d(p').$$

Spectrum of the operator T_b in ces(p)

Direct calculation: Every $\lambda \in \mathbb{D}$ is **eigenvalue** of S_p^* with eigenvector $x_\lambda = (1, \lambda, \lambda^2, \lambda^3, \dots) \in d(p')$.

$\Rightarrow$ $\mathbb{D} \subseteq \sigma_{\text{pt}}(S_p^*) = \sigma(S_p)$.

$\Rightarrow$ For all $b = \sum_{n=0}^{\infty} b_n e_n \in \ell^1$ we have

$$T_b = \sum_{n=0}^{\infty} b_n T_{e_n} = \sum_{n=0}^{\infty} b_n S_p^n \quad \text{(absolute conv. in } \|\cdot\|_{\text{op}}\text{)}.$$
$$\Rightarrow T_b^* = \sum_{n=0}^{\infty} b_n (S_p^*)^n \quad \text{(absolute conv. in } \|\cdot\|_{\text{op}}\text{)}.$$

Since $(S_p^*)^n x_\lambda = \lambda^n x_\lambda, \forall n \in \mathbb{N}$, implies $T_b^* x_\lambda = \varphi_b(\lambda) x_\lambda$, we have

$$\varphi_b(\lambda) \in \sigma(T_b^*) = \sigma(T_b), \qquad \forall \lambda \in \mathbb{D}.$$

This implies the reverse inclusion to yield:

Proposition 2.

Let $1 < p < \infty$. Then, for each $b \in \ell^1$, we have

$$\sigma(T_b) = \varphi_b(\overline{\mathbb{D}}) = \left\{\varphi_b(z) : z \in \overline{\mathbb{D}}\right\}.$$

Wiener's theorem for the B-algebra $\mathscr{R}$

If $\varphi_b(z) \neq 0$ for all $z \in \overline{\mathbb{D}}$, then
$$z \mapsto 1/\varphi_b(z), \quad z \in \overline{\mathbb{D}}, \text{ belongs to } \mathscr{R}.$$

Then $(1/\varphi_b) = \varphi_c$ for some $c \in \ell^1$. This yields

Fact 1: $\mathscr{M}_p$ is an **inverse-closed** subalgebra of $\mathscr{L}_p$.

That is, if $T_b - \lambda I$ (with $b \in \ell^1$) is invertible in $\mathscr{L}_p$, then
$[(T_b - \lambda I)^{-1} : \mathrm{ces}(p) \to \mathrm{ces}(p)] = T_c$ for some $c \in \ell^1$.

Fact 2: The formula $\sigma(T_b) = \varphi_b(\overline{\mathbb{D}}) = \{\varphi_b(z) : z \in \overline{\mathbb{D}}\}$ implies that
$T_b \in \mathscr{L}$ **fails** to be compact $\forall b \in \ell^1 \setminus \{0\}$, $1 < p < \infty$.

Thank you for your attention!