The regularity of roots of smooth polynomials

Armin Rainer[1] (University of Vienna)
Joint work with Adam Parusiński (Université de Nice)

Banach Center, Bedlewo, July 2, 2018

[1]Funded by Austrian Science Fund (FWF), Project P 26735-N25

Perturbation theory for polynomials

The problem

Consider a monic complex polynomial

$$P_a(x)(Z) = P_{a(x)}(Z) = Z^n + \sum_{j=1}^{n} a_j(x) Z^{n-j},$$

where the coefficients $a_j : \mathbb{R}^m \supseteq U \to \mathbb{C}$, $j = 1, \ldots, n$, possess some smoothness (C^ω, C^∞ , C^k, etc.).

How regular can we parameterize the roots λ_j of P_a?

Table of Contents

Hyperbolic polynomials: all roots are real

Bronshtein's theorem [Bronshtein '79], [Wakabayashi '86], [Parusiński, R. 14]

If $a \in C^{n-1,1}(\mathbb{R}^m, \mathbb{R}^n)$ then any continuous root λ is $C^{0,1}$.

Note: $\lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_n(x)$ are C^0 if $a(x)$ is C^0.

Hyperbolic polynomials: all roots are real

Bronshtein's theorem [Bronshtein '79], [Wakabayashi '86], [Parusiński, R. 14]

If $a \in C^{n-1,1}(\mathbb{R}^m, \mathbb{R}^n)$ then any continuous root λ is $C^{0,1}$.

Note: $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x)$ are C^0 if $a(x)$ is C^0.

Further Results

- If $a \in C^\omega(\mathbb{R}, \mathbb{R}^n)$ then λ_i can be chosen C^ω. [Rellich '37]
- If $a \in C^n(\mathbb{R}, \mathbb{R}^n)$ then λ_i can be chosen C^1. [Colombini, Orrú, Pernazza '12], [Parusiński, R. '14]
- If $a \in C^{2n}(\mathbb{R}, \mathbb{R}^n)$ then λ_i can be chosen twice differentiable. [Alekseevsky, Kriegl, Losik, Michor '98], [Colombini, Orrú, Pernazza '12]

Hyperbolic polynomials: all roots are real

Bronshtein's theorem [Bronshtein '79], [Wakabayashi '86], [Parusiński, R. 14]

If $a \in C^{n-1,1}(\mathbb{R}^m, \mathbb{R}^n)$ then any continuous root λ is $C^{0,1}$.

Note: $\lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_n(x)$ are C^0 if $a(x)$ is C^0.

Further Results

- If $a \in C^\omega(\mathbb{R}, \mathbb{R}^n)$ then λ_i can be chosen C^ω. [Rellich '37]
- If $a \in C^n(\mathbb{R}, \mathbb{R}^n)$ then λ_i can be chosen C^1. [Colombini, Orrú, Pernazza '12], [Parusiński, R. '14]
- If $a \in C^{2n}(\mathbb{R}, \mathbb{R}^n)$ then λ_i can be chosen twice differentiable. [Alekseevsky, Kriegl, Losik, Michor '98], [Colombini, Orrú, Pernazza '12]
- If $a \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ then $\lambda_i \notin C^{1,\omega}$ for any modulus of continuity ω. [Bony, Broglia, Colombini, Pernazza '06]
- If $a \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ is definable then λ_i can be chosen C^∞. [R. '11]

We have better behavior if no two (distinct) roots have infinite order of contact:

Results

- If $a \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ and no two roots meet of infinite order, then λ_i can be chosen C^∞. [Alekseevsky, Kriegl, Losik, Michor '98]

We have better behavior if no two (distinct) roots have infinite order of contact:

Results

- If $a \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ and no two roots meet of infinite order, then λ_i can be chosen C^∞. [Alekseevsky, Kriegl, Losik, Michor '98]
- If $a \in C^\omega(\mathbb{R}^m, \mathbb{R}^n)$ then there is a locally finite composite π of blow-ups such that the roots of $P_a \circ \pi$ admit C^ω-parameterizations locally. [Kurdyka, Paunescu '08]

We have better behavior if no two (distinct) roots have infinite order of contact:

Results

- If $a \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ and no two roots meet of infinite order, then λ_i can be chosen C^∞. [Alekseevsky, Kriegl, Losik, Michor '98]
- If $a \in C^\omega(\mathbb{R}^m, \mathbb{R}^n)$ then there is a locally finite composite π of blow-ups such that the roots of $P_a \circ \pi$ admit C^ω-parameterizations locally. [Kurdyka, Paunescu '08]
- There is a quasianalytic version of the latter result. [Chaumat, Chollet '04, 1-dim], [Nowak '11], [R. 2011].

Application: Weakly hyperbolic Cauchy problem

Theorem (Bronshtein '80)

Let $x = (x_0, x') \in U \subseteq \mathbb{R}^{1+m}$*,* U *a neighborhood of the origin, and consider*

$$P(x,D)u(x) = \sum_{|\alpha|\leq n} a_\alpha(x) D^\alpha u(x) = f(x),$$

$$D^j_{x_0} u(x)\big|_{x_0=0} = g_j(x'), \quad 0 \leq j \leq n-1, \quad D_{x_j} = -i\partial_{x_j},$$

where $a_{(n,0,\dots,0)} \neq 0$ *and the differential operator* P *is hyperbolic relative to* x_0*, i.e., for all* $(x,\xi') \in U \times \mathbb{R}^m$ *the principal symbol* $P_n(x,\xi)$ *is hyperbolic in* ξ_0*.*
Then the system is locally solvable in the Gevrey class G^s *for* $1 \leq s \leq \frac{n}{n-1}$*.*

$f \in G^s(\Omega)$ if and only if $f \in C^\infty(\Omega)$ and

$$\forall K \subseteq \Omega \text{ cp. } \exists C > 0\ \forall x \in K\ \forall \alpha : |f^{(\alpha)}(x)| \leq C^{|\alpha|+1} |\alpha|!^s$$

Table of Contents

Motivation from PDEs

Theorem (Spagnolo 00)

Let $I \times U \subseteq \mathbb{R} \times \mathbb{R}^m$ be open. Consider the pseudodifferential $n \times n$ system

$$u_t + iA(t, D_x)u + B(t, D_x)u = f(t, x), \quad (t, x) \in I \times U,$$

where $A(t, \xi)$, $B(t, \xi)$ are matrix symbols of order 1, resp. 0, and $A(t, \xi)$ is homogeneous of degree 1 in ξ for $|\xi| \geq 1$.
Assume that the eigenvalues of $A(t, \xi)$ admit a parameterization $\lambda_j(t, \xi)$ such that

1. *$\lambda_j(t, \xi)$ are absolutely continuous (AC) in t uniformly w.r.t. ξ*
2. *$\forall \xi$ either $\operatorname{Im} \lambda_j(t, \xi) \geq 0$ $\forall t, j$ or $\operatorname{Im} \lambda_j(t, \xi) \leq 0$ $\forall t, j$*

Then the system is locally solvable in the Gevrey class G^s for $1 \leq s \leq \frac{n}{n-1}$ and semi-globally solvable in G^s for $1 < s < \frac{n}{n-1}$.

More precisely, $A \in [C^\infty(I, S^1(\mathbb{R}^m))]^{n \times n}$, $B \in [C^0(I, S^0(\mathbb{R}^m))]^{n \times n}$, where $S^k(\mathbb{R}^m)$ is the set of all $a \in C^\infty(\mathbb{R}^m)$ such that for all $\alpha \in \mathbb{N}^m$ there exists $C_\alpha > 0$ such that

$$|\partial^\alpha a(\xi)| \leq C_\alpha \langle \xi \rangle^{k-|\alpha|} \quad \text{for all } \xi \in \mathbb{R}^m, \quad \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}.$$

Spagnolo's question

Open problem

Does a C^∞-curve $\mathbb{R} \ni t \mapsto P_a(t)$ of complex monic polynomials admit a parameterization of its roots that is locally absolutely continuous?

If so is this possible in a uniform way?

Spagnolo's question

Open problem

Does a C^∞-curve $\mathbb{R} \ni t \mapsto P_a(t)$ of complex monic polynomials admit a parameterization of its roots that is locally absolutely continuous?

If so is this possible in a uniform way?

- Note that in the absence of hyperbolicity we cannot expect that the roots are locally Lipschitz, e.g., $Z^2 = t$, $t \in \mathbb{R}$.

Spagnolo's question

Open problem

Does a C^∞-curve $\mathbb{R} \ni t \mapsto P_a(t)$ of complex monic polynomials admit a parameterization of its roots that is locally absolutely continuous?

If so is this possible in a uniform way?

- Note that in the absence of hyperbolicity we cannot expect that the roots are locally Lipschitz, e.g., $Z^2 = t$, $t \in \mathbb{R}$.
- If $a \in C^0(\mathbb{R}, \mathbb{C}^n)$ then the λ_i can be chosen C^0. [Kato 1976]

Radicals

Theorem (Ghisi, Gobbino '13)

Let $k \in \mathbb{N}_{\geq 1}$, $\alpha \in (0,1]$, *and let* $I \subseteq \mathbb{R}$ *be an open bounded interval. Assume that* $f \in C^0(I,\mathbb{R})$ *and there exists* $g \in C^{k,\alpha}(\overline{I},\mathbb{R})$ *such that*

$$|f|^{k+\alpha} = |g|.$$

Let p *be defined by* $\frac{1}{p} + \frac{1}{k+\alpha} = 1$. *Then we have* $f' \in L^p_w(I)$ *and*

$$\|f'\|_{p,w,I} \leq C(k) \max\left\{ \left(\mathrm{Höld}_{\alpha,I}(g^{(k)})\right)^{\frac{1}{k+\alpha}} |I|^{\frac{1}{p}}, \|g'\|^{\frac{1}{k+\alpha}}_{L^\infty(I)} \right\}.$$

Radicals

Theorem (Ghisi, Gobbino '13)

Let $k \in \mathbb{N}_{\geq 1}$, $\alpha \in (0,1]$, and let $I \subseteq \mathbb{R}$ be an open bounded interval. Assume that $f \in C^0(I,\mathbb{R})$ and there exists $g \in C^{k,\alpha}(\overline{I},\mathbb{R})$ such that

$$|f|^{k+\alpha} = |g|.$$

Let p be defined by $\frac{1}{p} + \frac{1}{k+\alpha} = 1$. Then we have $f' \in L^p_w(I)$ and

$$\|f'\|_{p,w,I} \leq C(k) \max\left\{ \left(\text{Höld}_{\alpha,I}(g^{(k)})\right)^{\frac{1}{k+\alpha}} |I|^{\frac{1}{p}}, \|g'\|^{\frac{1}{k+\alpha}}_{L^\infty(I)} \right\}.$$

A measurable $u : I \to \mathbb{C}$ belongs to the weak Lebesgue space $L^p_w(I)$ if

$$\|u\|_{p,w,I} := \sup_{r \geq 0} r \, |\{x \in I : |u(x)| > r\}|^{\frac{1}{p}} < \infty.$$

For $1 \leq q < p < \infty$ we have $L^p(I) \subsetneq L^p_w(I) \subsetneq L^q(I) \subsetneq L^q_w(I)$.

Optimality

- The conclusion is optimal: f' is in general not in L^p. Indeed, for $g : (-1, 1) \to \mathbb{R}$, $g(t) = t$, we have

$$\left(\frac{|g'|}{|g|^{1-\frac{1}{k+\alpha}}} \right)^p = |t|^{-1},$$

which is not integrable near 0.

Optimality

- The conclusion is optimal: f' is in general not in L^p. Indeed, for $g : (-1, 1) \to \mathbb{R}$, $g(t) = t$, we have

$$\left(\frac{|g'|}{|g|^{1-\frac{1}{k+\alpha}}} \right)^p = |t|^{-1},$$

which is not integrable near 0.

- The assumptions are optimal. There exists a non-negative function $f : I \to \mathbb{R}$ which belongs to $C^{k,\beta}(\overline{I}) \cap C^\infty(I)$ for every $\beta < \alpha$, but not for $\beta = \alpha$, and whose non-negative $(k + \alpha)$-root has unbounded variation in I.

Table of Contents

The general case

$$P_{a(t)}(Z) = Z^n + \sum_{j=1}^n a_j(t) Z^{n-j}$$

Theorem (Parusiński, R., Ann. Sci. Éc. Norm. Supér. (4), to appear)

Let $(\alpha, \beta) \subseteq \mathbb{R}$ be a bounded open interval.

1. *If $a \in C^{n-1,1}([\alpha, \beta], \mathbb{C}^n)$ then any C^0-root λ is AC.*

The general case

$$P_{a(t)}(Z) = Z^n + \sum_{j=1}^n a_j(t) Z^{n-j}$$

Theorem (Parusiński, R., Ann. Sci. Éc. Norm. Supér. (4), to appear)

Let $(\alpha, \beta) \subseteq \mathbb{R}$ be a bounded open interval.

1. *If $a \in C^{n-1,1}([\alpha, \beta], \mathbb{C}^n)$ then any C^0-root λ is AC.*
2. *$\lambda' \in L^p(I)$ for every $1 \leq p < \frac{n}{n-1}$. In particular, $\lambda \in W^{1,p}(I)$.*

The general case

$$P_{a(t)}(Z) = Z^n + \sum_{j=1}^n a_j(t) Z^{n-j}$$

Theorem (Parusiński, R., Ann. Sci. Éc. Norm. Supér. (4), to appear)

Let $(\alpha, \beta) \subseteq \mathbb{R}$ be a bounded open interval.

1. *If $a \in C^{n-1,1}([\alpha, \beta], \mathbb{C}^n)$ then any C^0-root λ is AC.*
2. *$\lambda' \in L^p(I)$ for every $1 \leq p < \frac{n}{n-1}$. In particular, $\lambda \in W^{1,p}(I)$.*
3. *Uniformity:*

$$\|\lambda'\|_{L^p((\alpha,\beta))} \leq C(n,p) \max\{1, (\beta - \alpha)^{1/p}\} \max_{1 \leq j \leq n} \|a_j\|_{C^{n-1,1}([\alpha,\beta])}^{1/j}.$$

Remarks

- The examples for radicals show that this result is optimal among Lebesgue spaces.

Remarks

- The examples for radicals show that this result is optimal among Lebesgue spaces.
- The proof is elementary.

Remarks

- The examples for radicals show that this result is optimal among Lebesgue spaces.
- The proof is elementary.
- Is $\lambda' \in L_w^{\frac{n}{n-1}}((\alpha, \beta))$? The L_w^p-quasinorm is not σ-additive:

  in general $\quad \|f\|_{p,w,I}^p \neq \sum_j \|f\|_{p,w,I_j}^p \quad$ if $I = \bigsqcup_j I_j$ (countable)

Remarks

- The examples for radicals show that this result is optimal among Lebesgue spaces.
- The proof is elementary.
- Is $\lambda' \in L_w^{\frac{n}{n-1}}((\alpha, \beta))$? The L_w^p-quasinorm is not σ-additive:

  in general $\quad \|f\|_{p,w,I}^p \neq \sum_j \|f\|_{p,w,I_j}^p \quad$ if $I = \bigsqcup_j I_j$ (countable)

- Are there scale invariant estimates which could replace the above bound?

Remarks

- The examples for radicals show that this result is optimal among Lebesgue spaces.
- The proof is elementary.
- Is $\lambda' \in L_w^{\frac{n}{n-1}}((\alpha, \beta))$? The L_w^p-quasinorm is not σ-additive:

$$\text{in general} \quad \|f\|_{p,w,I}^p \neq \sum_j \|f\|_{p,w,I_j}^p \quad \text{if } I = \bigsqcup_j I_j \text{ (countable)}$$

- Are there scale invariant estimates which could replace the above bound?
- What happens if the coefficients depend on several variables?

Table of Contents

Multiparameter perturbation

Theorem (Parusiński, R.)

Let $\Omega \subseteq \mathbb{R}^m$ be a bounded open Lipschitz domain. Let P_a be a monic polynomial of degree n with coefficients $a_j \in C^{n-1,1}(\overline{\Omega})$, $j = 1, \dots, n$. Let $\lambda \in C^0(V)$ be a root of P_a on an open subset $V \subseteq \Omega$. Then λ belongs to the Sobolev space $W^{1,p}(V)$ for every $1 \le p < n/(n-1)$. The distributional gradient $\nabla\lambda$ satisfies

$$\|\nabla\lambda\|_{L^p(V)} \le C(m,n,p,\Omega) \max_{1 \le j \le n} \|a_j\|^{1/j}_{C^{n-1,1}(\overline{\Omega})}.$$

Multiparameter perturbation

Theorem (Parusiński, R.)

Let $\Omega \subseteq \mathbb{R}^m$ be a bounded open Lipschitz domain. Let P_a be a monic polynomial of degree n with coefficients $a_j \in C^{n-1,1}(\overline{\Omega})$, $j = 1, \ldots, n$. Let $\lambda \in C^0(V)$ be a root of P_a on an open subset $V \subseteq \Omega$. Then λ belongs to the Sobolev space $W^{1,p}(V)$ for every $1 \le p < n/(n-1)$. The distributional gradient $\nabla\lambda$ satisfies

$$\|\nabla\lambda\|_{L^p(V)} \le C(m,n,p,\Omega) \max_{1 \le j \le n} \|a_j\|^{1/j}_{C^{n-1,1}(\overline{\Omega})}.$$

Here we must impose continuity of λ! E.g. $Z^2 = w$, $w \in \mathbb{C}$, has no solution that is continuous (or $W^{1,1}$) near the origin.

Are the roots in some suitable bigger space?

A natural candidate for a space

- larger than $W^{1,1}$, and
- allowing for jump type discontinuities

is the space BV of functions of bounded variation.

Are the roots in some suitable bigger space?

A natural candidate for a space

- larger than $W^{1,1}$, and
- allowing for jump type discontinuities

is the space BV of functions of bounded variation.

Do the roots of monic polynomials with smooth coefficients in several variables admit representations by BV-functions?

BV-functions

- Let $U \subseteq \mathbb{R}^m$ be open. A function $f \in L^1(U)$ is a BV-function in U if its distributional derivative is a finite Radon measure, i.e.,

$$\int_U f \partial_i \varphi \, dx = - \int_U \varphi \, dD_i f, \quad \forall \varphi \in C_c^\infty(U), i = 1, \ldots, m,$$

for some $\mathbb{R}^m$-valued measure $Df = (D_1 f, \ldots, D_m f)$ in U.
- $W^{1,1}(U) \subsetneq BV(U)$ with $Df = \nabla f \, \mathcal{L}^m$.
- $f \in BV(U)$ if and only if

$$\mathrm{Var}(f, U) := \sup \left\{ \int_U f \operatorname{div} \varphi \, dx : \varphi \in C_c^\infty(U, \mathbb{R}^m), \ \|\varphi\|_\infty \leq 1 \right\} < \infty.$$

Then $\mathrm{Var}(f, U) = |Df|(U)$.
- The variation $\mathrm{Var}(f, U)$ is lsc:

$$\mathrm{Var}(f, U) \leq \liminf_{k \to \infty} \mathrm{Var}(f_k, U), \quad \text{for } f_k \to f \text{ in } L^1_{\mathrm{loc}}.$$

- $\left(BV(U), \|f\|_{BV} = \|f\|_{L^1(U)} + |Df|(U)\right)$ is a Banach space.

Approximate discontinuities and jump points

- $f \in L^1_{\mathrm{loc}}(U, \mathbb{R}^\ell)$ has an approximate limit at $x \in U$ if

$$\exists z \in \mathbb{R}^\ell : \lim_{r \downarrow 0} \fint_{B_r(x)} |f(y) - z| \, dy = 0.$$

  The approximate discontinuity set S_f is the set of $x \in U$, where this is not true.

- $x \in U$ is an approximate jump point of f if

$$\exists a^\pm \in \mathbb{R}^\ell, a^- \neq a^+, \nu \in \mathbb{S}^{m-1} : \lim_{r \downarrow 0} \fint_{B_r^\pm(x,\nu)} |f(y) - a^\pm| \, dy = 0,$$

$$\text{where} \quad B_r^\pm(x,\nu) := \{y \in B_r(x) : \pm\langle y - x, \nu \rangle > 0\}.$$

- The set J_f of approximate jump points is a Borel subset of S_f, and $f^\pm : J_f \to \mathbb{R}^\ell$ and $\nu_f : J_f \to \mathbb{S}^{m-1}$ defined by $(f^+(x), f^-(x), \nu_f(x)) := (a^+, a^-, \nu)$ are Borel functions.

Decomposition of Df

- Let $f \in BV(U, \mathbb{R}^\ell)$. The Lebesgue decomposition gives

$$Df = D^a f + D^s f;$$

$D^a f$ is the absolutely continuous and $D^s f$ is the singular part of Df w.r.t. $\mathcal{L}^m$.

- We have ($Df = 0$ on $S_f \setminus J_f$ which is $\mathcal{H}^{m-1}$-negligible.)

$$\begin{aligned} Df &= D^a f + D^j f + D^c f \quad \text{where} \\ D^j f &:= D^s f \llcorner J_f \quad \ldots \text{ the jump part of } Df \\ D^c f &:= D^s \llcorner (U \setminus S_f) \quad \ldots \text{ the Cantor part of } Df. \end{aligned}$$

A BV-function f is a SBV-function if $D^c f = 0$.

- By the Federer–Vol'pert theorem

$$\begin{aligned} D^a f &= \nabla f \, \mathcal{L}^m, \\ D^j f &= (f^+ - f^-) \otimes \nu_f \, \mathcal{H}^{m-1} \llcorner J_f. \end{aligned}$$

Discontinuity set of radicals of smooth functions

Can we choose the discontinuity set of the roots in a good way?

Discontinuity set of radicals of smooth functions

Can we choose the discontinuity set of the roots in a good way?

Lemma

Let n, m, k be integers with $n \geq 2$, $m \geq 2$, and $k \geq m - 1$. Let $U \subseteq \mathbb{R}^m$ be open and $f \in C^{k,1}(U, \mathbb{C})$. Then there exist a C^k-hypersurface $E \subseteq U$ (possibly empty) and functions $\lambda_j \in C^0(U \setminus E, \mathbb{C})$, $j = 1, \ldots, n$, s.t.

$$Z^n - f = \prod_{j=1}^{n} (Z - \lambda_j).$$

Proof: Apply Sard's theorem to $\operatorname{sgn}(f) = f/|f| : U \setminus f^{-1}(0) \to \mathbb{S}^1$.

Discontinuity set of radicals of smooth functions

Can we choose the discontinuity set of the roots in a good way?

Lemma

Let n, m, k be integers with $n \geq 2$, $m \geq 2$, and $k \geq m-1$. Let $U \subseteq \mathbb{R}^m$ be open and $f \in C^{k,1}(U, \mathbb{C})$. Then there exist a C^k-hypersurface $E \subseteq U$ (possibly empty) and functions $\lambda_j \in C^0(U \setminus E, \mathbb{C})$, $j = 1, \ldots, n$, s.t.

$$Z^n - f = \prod_{j=1}^n (Z - \lambda_j).$$

Proof: Apply Sard's theorem to $\operatorname{sgn}(f) = f/|f| : U \setminus f^{-1}(0) \to \mathbb{S}^1$.

But in general we cannot choose the radicals of f in such a way that its discontinuity set has finite $\mathcal{H}^{m-1}$-measure!

Example

There exists $f \in C_c^\infty(\mathbb{R}^2)$ such that for any $\lambda : \mathbb{R}^2 \to \mathbb{C}$ with $\lambda^2 = f$ we have $\mathcal{H}^1(S_\lambda) = \infty$.

Level sets of the sign

Theorem

Let $k \in \mathbb{N}_+$, $\alpha \in (0,1]$, and set $s = k + \alpha$. Let $\Omega \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain and $f \in C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^{\ell+1})$, where $m \geq \ell \geq 1$. Then there is a constant $C = C(m, \ell, k, \alpha, \Omega)$ such that for each small $\varepsilon > 0$

$$\mathcal{H}^\ell\left(\left\{y \in \mathbb{S}^\ell : \int_{\operatorname{sgn}(f)^{-1}(y)} |f|^{\ell/s}\, d\mathcal{H}^{m-\ell} \geq \varepsilon^{-1}\, C\, \|f\|_{C^{k,\alpha}(\overline{\Omega})}^{\ell/s}\right\}\right) \leq \varepsilon.$$

In particular: the jump height of a suitable selection of a radical is integrable along its discontinuity set:

$$\int_{\operatorname{sgn}(f)^{-1}(y)} |f|^{1/n}\, d\mathcal{H}^{m-1} < \infty \quad \text{for } \mathcal{H}^1\text{-a.e. } y \in \mathbb{S}^1.$$

Proof: Optimal regularity of radicals and coarea formula;

$$\int_\Omega g(x)|J_\ell h(x)|\, dx = \int_{\mathbb{R}^\ell} \int_{h^{-1}(y)} g\, d\mathcal{H}^{m-\ell}\, dy, \quad \text{where } m \geq \ell \geq 1$$

$h \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^\ell)$ precisely represented, $p > \ell$ or $p \geq \ell = 1$, $g \geq 0$ measurable

Level sets of the norm

Theorem

Let $k \in \mathbb{N}_+$, $\alpha \in (0,1]$, *and set* $s = k + \alpha$. *Let* $\Omega \subseteq \mathbb{R}^m$ *be a bounded Lipschitz domain and* $f \in C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^{\ell+1})$, $f \not\equiv 0$. *Then there is a constant* $C = C(m, \ell, k, \alpha, \Omega)$ *such that for all* $0 < \varepsilon \leq 1$ *and all small* $\delta > 0$ *we have*

$$\left|\left\{y \in (0,\delta) : y^{1/s}\, \mathcal{H}^{m-1}(|f|^{-1}(y)) \geq \varepsilon^{-1} C \, \|f\|^{1/s}_{C^{k,\alpha}(\overline{\Omega})}\right\}\right| \leq \varepsilon\delta.$$

Proof: Coarea formula and optimal regularity of radicals.

Radicals of bounded variation

Theorem (Parusiński, R. '18)

Let $r \in \mathbb{R}_{>0}$, $m \in \mathbb{N}_{\geq 2}$. Let $k \in \mathbb{N}$, $\alpha \in (0,1]$ be s.t. $k + \alpha \geq \max\{r, m\}$. Let $\Omega \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain. Let $f \in C^{k,\alpha}(\overline{\Omega})$. Then there exists a solution $\lambda \in SBV(\Omega)$ of the equation $Z^r = f$ s.t.

$$\|\lambda\|_{BV(\Omega)} \leq C(m,k,\alpha,\Omega)\,\|f\|^{1/r}_{C^{k,\alpha}(\overline{\Omega})}.$$

There is a C^k-hypersurface $E \subseteq \Omega$ (possibly empty) such that λ is continuous on $\Omega \setminus E$ and satisfies $\nabla\lambda \in L^p_w(\Omega \setminus \overline{E})$ for $p = \frac{r}{r-1}$. We have

$$\|\nabla\lambda\|_{L^p_w(\Omega\setminus\overline{E})} \leq C(m,k,\alpha,\Omega)\,\|f\|^{1/r}_{C^{k,\alpha}(\overline{\Omega})},$$

and

$$\int_E |f|^{1/r}\, d\mathcal{H}^{m-1} \leq C(m,k,\alpha,\Omega)\,\|f\|^{1/r}_{C^{k,\alpha}(\overline{\Omega})}.$$

All discontinuities are jump discontinuities.

Roots of bounded variation

Theorem (Parusiński, R. '18)

For all integers $n, m \geq 2$ *there is an integer* $k = k(n,m) \geq \max(n,m)$ *s.t. the following holds. Let* $\Omega \subseteq \mathbb{R}^m$ *be a bounded Lipschitz domain and*

$$P_a(x)(Z) = P_{a(x)}(Z) = Z^n + \sum_{j=1}^n a_j(x) Z^{n-j}, \quad x \in \Omega,$$

a monic polynomial with coefficients $a = (a_1, \ldots, a_n) \in C^{k-1,1}(\overline{\Omega}, \mathbb{C}^n)$*. Then the roots of* P_a *admit a parameterization* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *by* SBV*-functions on* Ω *such that*

$$\|\lambda\|_{BV(\Omega)} \leq C(n,m,\Omega) \max\left\{1, \|a\|_{L^\infty(\Omega)}\right\} \max\left\{1, \|a\|_{C^{k-1,1}(\overline{\Omega})}\right\}.$$

There is a finite collection of C^{k-1}*-hypersurfaces* E_j *in* Ω *such that* λ *is continuous in the complement of* $E := \bigcup_j E_j$*. All discontinuities of* λ *are jump discontinuities. For all* $1 \leq p < \frac{n}{n-1}$*,*

$$\|\lambda\|_{W^{1,p}(\Omega \setminus \overline{E})} \leq C(n,m,p,\Omega) \max_{1 \leq j \leq n} \|a_j\|^{1/j}_{C^{n-1,1}(\overline{\Omega})}.$$

Idea of the proof in the general case

- In [Parusiński, R., Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), '16] we proved formulas for the roots of the universal polynomial P_a, $a \in \mathbb{C}^n$. The roots are expressed as finite sums of functions analytic in radicals of local coordinates on a resolution space (a blowing up of $\mathbb{C}^n$).

Idea of the proof in the general case

- In [Parusiński, R., Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), '16] we proved formulas for the roots of the universal polynomial P_a, $a \in \mathbb{C}^n$. The roots are expressed as finite sums of functions analytic in radicals of local coordinates on a resolution space (a blowing up of $\mathbb{C}^n$).
- We choose parameterizations of the involved radicals. In this way we obtain SBV-parameterizations of these summands.

Idea of the proof in the general case

- In [Parusiński, R., Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), '16] we proved formulas for the roots of the universal polynomial P_a, $a \in \mathbb{C}^n$. The roots are expressed as finite sums of functions analytic in radicals of local coordinates on a resolution space (a blowing up of $\mathbb{C}^n$).
- We choose parameterizations of the involved radicals. In this way we obtain SBV-parameterizations of these summands.
- But then a new difficulty arises which comes from the fact that these summands are defined only locally on the resolution space. (Actually, they cannot be defined neither globally nor canonically.) We solve this problem by cutting and pasting these locally defined summands which introduces new discontinuities. In order to stay in the class SBV we must ascertain integrability of the new jumps along these discontinuities. This is again based on a consequence of Ghisi and Gobbino's result for radicals and the coarea formula.

Open problems

- The uniform bound for $\|\lambda\|_{BV}$ is not scale-invariant.

Open problems

- The uniform bound for $\|\lambda\|_{BV}$ is not scale-invariant.
- The degree of differentiability k is not sharp; it is sharp in the radical case.

Open problems

- The uniform bound for $\|\lambda\|_{BV}$ is not scale-invariant.
- The degree of differentiability k is not sharp; it is sharp in the radical case.
- The method of our proof is local. This forces us to deal with the global monodromy by cutting and pasting the local choices of the roots. It introduces additional discontinuities some of which are perhaps not necessary. It would be interesting to have a global understanding of the monodromy and the discontinuities it necessitates.

Thank you for your attention!