Dense quasi-free topological subalgebras of the Toeplitz algebra

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Paweł Domański Memorial Conference
Będlewo, 2018

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   - $v = T_z$, where z is the coordinate on $\mathbb{T}$.

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   - $\mathcal{T}_{\mathrm{alg}} \cong \mathbb{C}[v, u] \cong \mathbb{C}[v] \otimes \mathbb{C}[u]$ as a vector space.

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Definition (N. C. Phillips, 1991)

The algebra of smooth compact operators is

$$\mathscr{K}_{\mathrm{smth}} = \Big\{ a = (a_{ij})_{i,j \in \mathbb{Z}_+} : a_{ij} \in \mathbb{C},\ \|a\|_n = \sum_{i,j} (1+i+j)^n |a_{ij}| < \infty \ \forall n \in \mathbb{Z}_+ \Big\}$$

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- Topologize $\mathcal{T}_{\mathrm{smth}}$ via the above isomorphism.
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- Algebras of countable dimension equipped with the strongest locally convex topology (e.g., $\mathbb{C}[z]$, $\mathbb{C}[z, z^{-1}]$, M_∞, $\mathcal{T}_{\mathrm{alg}}$ ...)

Algebra extensions

- $S = \widehat{\otimes}$-algebra
- An extension of S is an exact sequence

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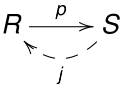

- (1) is admissible $\iff$ there exists a continuous linear map $j\colon S \to R$ such that $pj = \mathbf{1}_S$.
- (1) is a square-zero extension if $I^2 = 0$.

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Other equivalent definitions

- The A-bimodule $\Omega^1 A$ of noncommutative differential 1-forms is projective.
- $\operatorname{bidim} A \leq 1$.
- etc. ...

Examples of quasi-free algebras

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7. (Helemskii, 1972): A commutative Banach algebra with infinite spectrum is not quasi-free.

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Goal:

Construct a family $\{\mathcal{T}_{P,Q}\}$ of dense locally convex subalgebras of $\mathcal{T}$, and give a sufficient condition for $\mathcal{T}_{P,Q}$ to be quasi-free.

Power series algebras

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$$\lambda(P) = \left\{ a = (a_k) \in \mathbb{C}^{\mathbb{Z}_+} : \|a\|_p = \sum_{k=0}^{\infty} |a_k| p_k < \infty \quad \forall\, p \in P \right\}.$$

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- $\lambda(z, P)$ is a $\widehat{\otimes}$-algebra.

Examples

1. $P = \{p^{(k)} : k \in \mathbb{N}\}$, where $p^{(k)} = (1, \ldots, 1, 0, 0, \ldots)$. Then
$$\lambda(z, P) = \mathbb{C}[[z]].$$
2. $P = \{p \in (0, +\infty)^{\mathbb{Z}_+} : p_0 = 1\}$. Then
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equipped with the strongest locally convex topology.
3. $P = \{p^{(r)} : 0 < r < R\}$, where $p_n^{(r)} = r^n$. Then
$$\lambda(z, P) \cong \mathscr{O}(\mathbb{D}_R),$$
the algebra of holomorphic functions on $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$.
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$$\lambda(z, P) \cong \{f \in C^\infty(\mathbb{T}) : f \text{ extends to a holomorphic function on } \mathbb{D}\}.$$

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Suppose that P and Q are monotone weighted sets such that $P * P \prec P$ and $Q * Q \prec Q$. Then $\mathcal{T}_{P,Q}$ is quasi-free.

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Corollary

$\mathcal{T}_{\mathrm{smth}}$ and $\mathcal{T}_{\mathrm{hol}}$ are quasi-free.