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Complex symmetric composition operators on spaces of analytic functions

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Complex symmetric operators

T – bounded linear operator on a separable complex Hilbert space H.

A conjugation is an anti-linear operator $C\colon H \to H$ such that

- $C^2 = I$
- C is isometric

T is C-symmetric if $T = CT^*C$

T is complex symmetric if there exists a conjugation C such that T is C-symmetric.

Equivalently
T is complex symmetric if there exists an orthonormal basis in H in which T has a self-transpose matrix representation.

Main quests

- given an operator resolve wheter it is complex symmetric
- indicate a concrete conjugation with respect to which it is complex symmetric
- what properties possess complex symmetric operators

CS operators – example

$L^2[0,1]$ with an orthonormal basis $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$

$T\colon L^2[0,1]\to L^2[0,1]$ – the Volterra operator

$$Tf(x)=\int_0^x f(t)\,\mathrm{d}t,\quad f\in L^2[0,1],\ x\in[0,1]$$

$$T=CT^*C=\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \frac{i}{4\pi} & 0 & -\frac{i}{4\pi} & 0 & 0 & \cdots \\ \cdots & 0 & \frac{i}{2\pi} & -\frac{i}{2\pi} & 0 & 0 & \cdots \\ \cdots & -\frac{i}{4\pi} & -\frac{i}{2\pi} & \frac{1}{2} & \frac{i}{2\pi} & \frac{i}{4\pi} & \cdots \\ \cdots & 0 & 0 & \frac{i}{2\pi} & -\frac{i}{2\pi} & 0 & \cdots \\ \cdots & 0 & 0 & \frac{i}{4\pi} & 0 & -\frac{i}{4\pi} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$Cf(x)=\overline{f(1-x)},\quad f\in L^2[0,1],\ x\in[0,1]$$

CS operators – more examples

- all 2×2 complex matrices
- normal operators
- compressed Toeplitz operators (compression of Toeplitz operator to the invariant subspace of unilateral shift on H^2)
- Hankel operators.

An operator T defined by a matrix

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 1 \end{bmatrix}, \quad |a| \neq |b|$$

is not complex symmetric.

History

S. R. Garcia and M. Putinar
Complex symmetric operators and applications
Trans. Amer. Math. Soc. 358 (2006) 1285–1315.

S. R. Garcia and C. Hammond
Which Weighted Composition Operators Are Complex Symmetric?
Oper. Theory Adv. Appl., vol. 236, Birkhuser/Springer Basel, 2014, 171–179.

P.S. Bourdon and S. Waleed Noor
Complex symmetry of invertible composition operators
J. Math. Anal. Appl. 429 (2015), 105–110.

Bergman spaces

$H(\mathbb{D})$ – holomorphic functions on $\{z \in \mathbb{C} : |z| < 1\}$

Bergman space

$f \in H(\mathbb{D})$

$$\|f\|_{A^2} = \left(\int_{\mathbb{D}} |f(z)|^2 dA(z)\right)^2 < \infty,$$

where $dA(z)$ is the normalized area measure on $\mathbb{D}$.

A^2 is a Hilbert space with the inner product

$$\langle f, g\rangle_{A^2} = \int_{\mathbb{D}} f(z)\overline{g(z)} dA(z).$$

Composition operators

$\varphi \in H(\mathbb{D})$, $\varphi \colon \mathbb{D} \to \mathbb{D}$,

$$C_\varphi f = f \circ \varphi, \quad f \in H(\mathbb{D})$$

Any composition operator is bounded on the Bergman space A^2.

Which symbols $\varphi \in H(\mathbb{D})$, $\varphi \colon \mathbb{D} \to \mathbb{D}$ generate complex symmetric composition operators on the Bergman space A^2.

General result

Theorem

If the composition operator $C_\varphi : A^2 \to A^2$ is complex symmetric then φ is either

- *an elliptic automorphism of $\mathbb{D}$ or*
- *has a Denjoy–Wolff point in $\mathbb{D}$.*

Denjoy–Wolff Theorem
If φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map of the disc into itself, then there exists a point $a \in \overline{\mathbb{D}}$ so that the iterates of φ converges to a uniformly on compact subsets of $\mathbb{D}$.

A point a from the above theorem is called a Denjoy–Wolff point of φ.

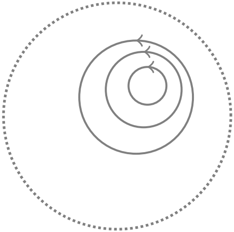
Linear fractional maps

For $\alpha \in \overline{D}$ define $\varphi_\alpha \colon \mathbb{D} \to \mathbb{D}$ by a formula

$$\varphi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \quad z \in \mathbb{D}.$$

A disc automorphism φ is called elliptic if there exists $|\lambda| = 1$ such that

$$\varphi = \varphi_\alpha \circ (\lambda \varphi_\alpha).$$



Order of a linear fractional map

Let φ be an automorphism of the form

$$\varphi = \varphi_\alpha \circ (\lambda \varphi_\alpha), \quad |\alpha| \leq 1.$$

- φ has finite order N if there exists N such that $\lambda^N = 1$
- φ has infinite order if no such integer exists.

Elliptic automorphism of infinite order

Theorem

Suppose φ is an elliptic automorphism of infinite order and is not a rotation. Then $C_\varphi : A^2 \to A^2$ is not complex symmetric.

Elliptic automorphism of finite order $N \geqslant 6$

Theorem

Suppose φ is an elliptic automorphism of finite order $N \geqslant 6$ and is not a rotation. Then $C_\varphi : A^2 \to A^2$ is not complex symmetric.

Elliptic automorphism of order 2

Theorem

- *If $\varphi \colon \mathbb{D} \to \mathbb{D}$ is a rotation, then $C_\varphi \colon A^2 \to A^2$ is a normal operator and thus complex symmetric.*
- *Suppose $\varphi = \varphi_\alpha \circ (\lambda \varphi_\alpha)$ is an elliptic automorphism of order two. Then $C_\varphi \colon A^2 \to A^2$ is complex symmetric.*

Theorem (S. R. Garcia and W. R. Wogen)

If an operator $T \colon H \to H$ on a Hilbert space H satisfies $p(T) = 0$ for some polynomial of degree 2 or less, then T is complex symmetric.

S. R. Garcia and W. R. Wogen
Some new classes of complex symmetric operators
Trans. Amer. Math. Soc. 362 (2010), 6065–6077.

Proofs: sweat and tears

- elements of linear dynamics (i.e., A^2-outer functions are cyclic in A^2)
- formula for the adjoint of C_{φ_α}

$$C_{\varphi_\alpha}^* = M_{K_\alpha} C_{\varphi_\alpha} M_{1/K_\alpha}^*$$

  where K_α is a reproducing kernel

$$K_\alpha(z) = \frac{1}{(1-\overline{\alpha}z)^2}, \quad \alpha, z \in \mathbb{D},$$

- $v_n \perp v_m$ if and only if $|n-m| \geq 3$, where

$$v_n := C_{\varphi_\alpha}^* z^n$$

P.R. Hurst
Relating composition operators on different weighted Hardy spaces
Arch. Math. 68 (1997) 503–513

Conclusion

If the composition operator $C_\varphi \colon A^2 \to A^2$ is complex symmetric then:

- φ has a Denjoy–Wolff point in the disc $\mathbb{D}$ or
- φ is a rotation or
- φ is an elliptic automorphism of finite order N, where $N = 2, 3, 4, 5$.

Open problem

Are composition operators generated by elliptic automorphism of order $3, 4$, or 5 complex symmetric?