On stability of isomorphisms between interpolation spaces

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Outline

Definition. A bounded linear operator $T\colon X \to Y$ between Banach spaces is said to be semi-Fredholm if $T(X)$ is closed in Y and at least one of the subspaces $\ker T$, $Y/T(X)$ is finite-dimensional. Then the index of T is given by

$$\operatorname{ind}(T) := \dim(\ker T) - \operatorname{codim} T(X)$$

If $\operatorname{ind}(T)$ is finite, T is called a Fredholm operator.

Example. A strictly singular perturbation of a Fredholm operator remains Fredhom and has the same index.

Remarks. (The Fredholm Alternative) If X is a Banach space and $K\colon X \to X$ is a compact operator, then for every $\lambda \neq 0$ exactly one of the following two exclusive statements is true:

(i) For every $y \in Y$ the equation $x - \lambda Kx = y$ has a unique solution;

(ii) The equation $x - \lambda Kx = 0$ has a non-trivial solution.

When (ii) is true, the equation $x - \lambda Kx = 0$ has a finite number of linearly independent solutions.

Introduction

- A mapping $F\colon \vec{\mathcal{B}} \to \mathcal{B}$ from the category $\vec{\mathcal{B}}$ of all couples of Banach spaces into the category $\mathcal{B}$ of all Banach spaces is said to be an interpolation functor if, for any couple $\vec{X} := (X_0, X_1)$, the Banach space $F(X_0, X_1)$ is intermediate with respect to $\vec{X}$ (i.e., $X_0 \cap X_1 \subset F(X_0, X_1) \subset X_0 + X_1$), and

$$T\colon F(X_0, X_1) \to F(Y_0, Y_1) \quad \text{for all } T\colon (X_0, X_1) \to (Y_0, Y_1);$$

- $T\colon (X_0, X_1) \to (Y_0, Y_1)$ means that $T\colon X_0 + X_1 \to Y_0 + Y_1$ is a linear operator such that the restrictions of T to the space X_j is a bounded operator from X_j to Y_j, for both $j = 0$ and $j = 1$.

- An operator $T\colon (X_0, X_1) \to (Y_0, Y_1)$ between Banach couples is said to be invertible whenever the restriction $T|_{X_j}\colon X_j \to Y_j$ is invertible (i.e., T is an isomorphism of X_j onto Y_j) for each $j \in \{0, 1\}$.

- The complex method Let $S := \{z \in \mathbb{C};\ 0 < \mathrm{Re} z < 1\}$ be an open strip on the plane. For a given $\theta \in (0,1)$ and any couple $\vec{X} = (X_0, X_1)$ we denote by $\mathcal{F}(\vec{X})$ the Banach space of all bounded continuous functions $f\colon \bar{S} \to X_0 + X_1$ on the closure $\bar{S}$ that are analytic on S, and

$$\mathbb{R} \ni t \mapsto f(j+it) \in X_j, \quad j = 0, 1$$

is a bounded continuous function, and equipped with the norm

$$\|f\|_{\mathcal{F}(\vec{\mathcal{X}})} = \max_{j=0,1} \sup_{t\in\mathbb{R}} \|f(j+it)\|_{X_j}.$$

The (lower) complex interpolation space $[\vec{X}]_\theta := \{f(\theta);\ f \in \mathcal{F}(\vec{X})\}$ and is equipped with the quotient norm.

- The real method For $\theta \in (0,1)$ and $p \in [1, \infty]$, $(X_0, X_1)_{\theta,p}$ is defined as the Banach space of all $x \in X_0 + X_1$ equipped with the norm

$$\|x\|_{\theta,p} = \left(\int_0^\infty \left[t^{-\theta} K(t, x; \vec{X})\right]^p \frac{dt}{t}\right)^{1/p},$$

where

$$K(t, x; \vec{X}) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1};\ x = x_0 + x_1\}, \quad t > 0.$$

Theorem

(I. Ya. Shneiberg, 1974) If $T\colon (X_0, X_1) \to (Y_0, Y_1)$ *is such that*

$$T\colon [X_0, X_1]_{\theta_*} \to [Y_0, Y_1]_{\theta_*}$$

is Fredholm for some $\theta_* \in (0,1)$*, then there exists* $\varepsilon > 0$ *such that*

$$T\colon [X_0, X_1]_{\theta} \to [Y_0, Y_1]_{\theta}$$

is Fredholm and the index is constant for all $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$*.*

- Remark. Let $\vec{X} = (X_0, X_1)$ be a complex Banach couple and $T\colon (X_0, X_1) \to (X_0, X_1)$ be an operator. If $0 \leqslant \alpha < \beta \leqslant 1$ and $T_\alpha := T|_{[\vec{X}]_\alpha}$ and $T_\beta := T|_{[\vec{X}]_\beta}$ are invertible, then the inverses T_α^{-1} and T_β^{-1} do not coincide on $X_0 \cap X_1$ in general.

- Example. The dilatation operator D_a ($a > 0$, $a \neq 1$) given by $D_a f(t) = f(at)$, $t > 0$ is bounded on $L^p = L^p(\mathbb{R}_+)$ for every $1 < p < \infty$ and

$$\sigma(D_a, L^p) = \left\{\lambda \in \mathbb{C};\ |\lambda| = a^{-1/p}\right\}.$$

If $|\lambda| = a^{-1/p}$, $p_0 < p < p_1$, then the operator $T := D_a - \lambda I$ is invertible on

$$L^{p_0} = [L^1, L^\infty]_\alpha, \quad L^{p_1} = [L^1, L^\infty]_\beta$$

with $\alpha = 1 - 1/p_0$ and $\beta = 1 - 1/p_1$ but T is not invertible on L^p.

- M. Zafran (1980) An operator $T\colon (X_0, X_1) \to (X_0, X_1)$ is said to have the uniqueness-of-resolvent property if

$$(T_\alpha - \lambda I)^{-1}|_{X_0 \cap X_1} = (T_\beta - \lambda I)^{-1}|_{X_0 \cap X_1}$$

for all α, $\beta \in [0, 1]$ and $\lambda \notin \sigma(T_\alpha) \cup \sigma(T_\beta)$.

- T. Ransford (1986) An operator $T\colon (X_0, X_1) \to (X_0, X_1)$ satisfies the local uniqueness-of-resolvent condition, if for all $\alpha \in (0, 1)$ and $\lambda \notin \sigma(T_\alpha)$, there exists a neighbourhood $U \subset (0, 1)$ of α such that $(T_\theta - \lambda I)^{-1}$ exists and

$$(T_\theta - \lambda I)^{-1} = (T_\alpha - \lambda I)^{-1}|_{X_0 \cap X_1}, \quad \theta \in U.$$

Albrecht and Müller (2000) If (X_0, X_1) is a complex Banach couple and $T\colon (X_0, X_1) \to (X_0, X_1)$ is such that $T_\alpha\colon [X_0, X_1]_\alpha \to [X_0, X_1]_\alpha$ is invertible for some $\alpha \in (0, 1)$, then there exists a neighbourhood $U \subset (0, 1)$ of α such that T_θ is invertible and T_θ^{-1} agrees with T_α^{-1} on $X_0 \cap X_1$ for any $\theta \in U$.

- A. P. Calderón (1983) If (Ω, Σ, μ) is a measure space and $T\colon L^p(\mu) \to L^p(\mu)$ is a bounded operator for $1 < p < \infty$, which is invertible for $p = 2$, then T is also invertible when $2 - \varepsilon < p < 2 + \varepsilon$, for some small $\varepsilon > 0$.

- A careful analysis of Calderón's proofs gives the compatibility of inverses, i.e., there exists some small $\varepsilon > 0$ such that for all $p, q \in (2 - \varepsilon, 2 + \varepsilon)$, the inverse T^{-1} considered on the space $L^p(\mu)$ is compatible with T^{-1} considered on $L^q(\mu)$ when both operators are restricted to $L^p(\mu) \cap L^q(\mu)$.

- J. Pipher and G. Verchota (1992) shown a useful application of Claderón's result for solvability of the Dirichlet problem with data in $L^p(\partial\Omega)$ for the biharmonic equation:

$$\Delta u = 0 \quad \text{in} \quad \Omega \quad \text{with} \quad u = f \quad \text{and} \quad \partial u/\partial n = g \text{ on } \partial\Omega,$$

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$.

- Let **Ban** be the class of all Banach spaces over the complex field. A mapping $\mathcal{X}\colon \mathbf{Ban} \to \mathbf{Ban}$ is called a pseudolattice, or a **pseudo-$\mathbb{Z}$-lattice**, if
(i) for every $B \in \mathbf{Ban}$ the space $\mathcal{X}(B)$ consists of B valued sequences $\{b_n\} = \{b_n\}_{n\in\mathbb{Z}}$ modelled on $\mathbb{Z}$;
(ii) whenever A is a closed subspace of B it follows that $\mathcal{X}(A)$ is a closed subspace of $\mathcal{X}(B)$;
(iii) there exists a positive constant $C = C(\mathcal{X})$ such that, for all A, $B \in \mathbf{Ban}$ and all bounded linear operators $T\colon A \to B$ and every sequence $\{a_n\} \in \mathcal{X}(A)$, the sequence $\{Ta_n\} \in \mathcal{X}(B)$ and satisfies the estimate

$$\|\{Ta_n\}\|_{\mathcal{X}(B)} \leqslant C\|T\|_{A\to B}\|\{a_n\}\|_{\mathcal{X}(A)};$$

(iv)

$$\|b_m\|_B \leqslant \|\{b_n\}\|_{\mathcal{X}(B)}$$

for each $m \in \mathbb{Z}$, all $\{b_n\} \in \mathcal{X}(B)$ and all Banach spaces B.

- For every Banach couple $\vec{B} = (B_0, B_1)$ and every Banach couple of pseudolattices $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$, let $\mathcal{J}(\vec{\mathcal{X}}, \vec{B})$ be the Banach space of all $B_0 \cap B_1$ valued sequences $\{b_n\}$ such that $\{e^{jn} b_n\} \in \mathcal{X}(B_j)$ $(j = 0, 1)$, equipped with the norm.

$$\|\{b_n\}\|_{\mathcal{J}(\vec{\mathcal{X}}, \vec{B})} = \max\left\{\|\{b_n\}\|_{\mathcal{X}_0(B_0)}, \|\{e^n b_n\}\|_{\mathcal{X}_1(B_1)}\right\}.$$

- Following Cwikel–Kalton–Milman–Rochberg (2002), for every s in the annulus $\mathbb{A} := \{z \in \mathbb{C};\, 1 < |z| < e\}$, we define the Banach space $\vec{B}_{\vec{X},s}$ to consist of all elements of the form $b = \sum_{n \in \mathbb{Z}} s^n b_n$ (convergence in $B_0 + B_1$ with $\{b_n\} \in \mathcal{J}(\vec{\mathcal{X}}, \vec{B})$, equipped with the norm

$$\|b\|_{\vec{B}_{\vec{\mathcal{X}},s}} = \inf\left\{\|\{b_n\}\|_{\mathcal{J}(\vec{\mathcal{X}}, \vec{B})};\ b = \sum_{n \in \mathbb{Z}} s^n b_n\right\}.$$

It is easy to check that the map $\vec{B} \mapsto \vec{B}_{\vec{\mathcal{X}},s}$ is an interpolation functor.

- A couple $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ of Banach pseudolattices, is said to be translation invariant if for any Banach space B,

$$\left\|\{S^k(\{b_n\}_{n\in\mathbb{Z}}\}\right\|_{\mathcal{X}_j(B)} = \left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{X}_j(B)}$$

  for all $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{X}_j(B)$, each $k \in \mathbb{Z}$ and $j \in \{0,1\}$, where is the left-shift operator defined by $S\{b_n\} = \{b_{n+1}\}$.
- $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ is said to be a rotation invariant Banach couple of pseudolattices whenever the rotation map

$$\{b_n\}_{n\in\mathbb{Z}} \mapsto \{e^{in\tau} b_n\}_{n\in\mathbb{Z}}$$

  is an isometry of $\mathcal{X}_j(B)$ onto itself for every real τ and every Banach space B.

Lemma

Let $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ be a Banach couple of rotation invariant pseudolattices. Then, for every Banach couple $\vec{B} = (B_0, B_1)$ and all $s \in \mathbb{A}$, we have

(i) *If $f \in \mathcal{F}_{\vec{\mathcal{X}}}(\vec{B})$, then $f(s) \in \vec{B}_{\vec{\mathcal{X}},|s|}$;*

(ii) *If $x \in \vec{B}_{\vec{\mathcal{X}},|s|}$, then there exists $f \in \mathcal{F}_{\vec{\mathcal{X}}}(\vec{B})$ such that $f(s) = x$;*

(iii) $\vec{B}_{\vec{\mathcal{X}},s} \cong \vec{B}_{\vec{\mathcal{X}},|s|}$.

Theorem

(I. Asekritova, N. Kruglyak & M. M.) Let $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ be a couple of rotational and translational invariant pseudolattices and let $\{F_\theta\}_{\theta \in (0,1)}$ be a family of interpolation functors defined by $F_\theta(X_0, X_1) := (X_0, X_1)_{\vec{\mathcal{X}}, e^\theta}$ for any Banach couple (X_0, X_1). Suppose also that $\vec{\mathcal{X}}$ is such that F_θ is regular functor and $F_\theta(X_0, X_1) = F_\theta(X_0^\circ, X_1^\circ)$ for any Banach couple (X_0, X_1). If an operator $T\colon (X_0, X_1) \to (Y_0, Y_1)$ is such that the operator

$$T|_{F_{\theta_*}(X_0,X_1)}\colon F_{\theta_*}(X_0, X_1) \to F_{\theta_*}(Y_0, Y_1)$$

is Fredholm. Then there exists $\varepsilon = \varepsilon(\theta_*, \tilde{\chi}) > 0$ such that for any $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$ the operator

$$T|_{F_\theta(X_0,X_1)}\colon F_\theta(X_0, X_1) \to F_\theta(Y_0, Y_1)$$

is also Fredholm and $\operatorname{ind}(T|_{F_\theta(X_0,X_1)}) = \operatorname{ind}(T|_{F_{\theta_*}(X_0,X_1)})$.

Theorem

Let $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ *be a Banach couple of translation invariant pseudolattices and let* $T\colon \vec{X} \to \vec{Y}$ *be an operator between complex Banach couples. Assume that* $T\colon \vec{X}_{\vec{\mathcal{X}},s} \to \vec{Y}_{\vec{\mathcal{X}},s}$ *is invertible for some* $s \in \mathbb{A}$*. Then* $T_\omega\colon \vec{X}_{\vec{\mathcal{X}},\omega} \to \vec{Y}_{\vec{\mathcal{X}},\omega}$ *is invertible for all* ω *in an open neighbourhood* $W = \{\omega \in \mathbb{A};\, |\omega - s| < r\}$ *of* s *in* $\mathbb{A}$ *with*

$$r = \big[2\delta(s)\big(1 + \|T\|_{\vec{X}\to\vec{Y}}\|T^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},s}\to\vec{X}_{\vec{\mathcal{X}},s}}\big)\big]^{-1},$$

where $\delta(s) = \max\big\{(|s|-1)^{-1},\, (e-|s|)^{-1}\big\}$*. Moreover the following upper estimate for the norm of* T_ω *holds,*

$$\|T_\omega^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},\omega}\to\vec{X}_{\vec{\mathcal{X}},\omega}} \leqslant 2\|T_s^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},s}\to\vec{X}_{\vec{\mathcal{X}},s}}, \quad \omega \in W.$$

Theorem

Let $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ be a couple of translation and rotation invariant pseudolattices and let $T\colon \vec{X} \to \vec{Y}$. Assume that $T_{\theta_*}\colon \vec{X}_{\vec{\mathcal{X}},e^{\theta_*}} \to \vec{Y}_{\vec{\mathcal{X}},e^{\theta_*}}$ is invertible for some $\theta_* \in (0,1)$. Then $T_\theta\colon \vec{X}_{\vec{\mathcal{X}},e^{\theta}} \to \vec{Y}_{\vec{\mathcal{X}},e^{\theta}}$ is invertible for all θ in an open neighbourhood $I = \{\theta \in (0,1);\ |\theta - \theta_*| < \varepsilon\}$ of θ_* with

$$\varepsilon = \left[2e\eta(\theta_*)\left(1 + \|T\|_{\vec{X}\to\vec{Y}}\|T^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},e^{\theta_*}} \to \vec{X}_{\vec{\mathcal{X}},e^{\theta_*}}}\right)\right]^{-1},$$

where $\eta(\theta_*) = \max\left\{(e^{\theta_*} - 1)^{-1}, (e - e^{\theta_*})^{-1}\right\}$. Moreover T_θ^{-1} agrees with $T_{\theta_*}^{-1}$ on $Y_0 \cap Y_1$ and

$$\|T_\theta^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},e^{\theta}} \to \vec{X}_{\vec{\mathcal{X}},e^{\theta}}} \leqslant 2\|T_{\theta_*}^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},e^{\theta_*}} \to \vec{X}_{\vec{\mathcal{X}},e^{\theta_*}}}, \quad \theta \in I.$$

Theorem

Let $\vec{\mathcal{X}}$ *be a couple of pseudolattices and let* $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$ *be complex Banach couples,* $T\colon \vec{X} \to \vec{Y}$ *and* $s \in \mathbb{A}$. *Assume that* $T\colon \vec{X}_{\vec{\mathcal{X}},s} \to \vec{Y}_{\vec{\mathcal{X}},s}$ *is invertible. Then, there exists an open neighborhood* $U \subset \mathbb{A}$ *of* s *such that, for all* $k \in \mathcal{F}_{\vec{\mathcal{X}}}(\vec{Y})$, *there exist analytic functions* $g\colon U \to \mathcal{F}_{\vec{\mathcal{X}}}(\vec{X})$ *and* $h\colon U \to \mathcal{F}_{\vec{\mathcal{X}}}(\vec{Y})$ *such that, for all* $\omega \in U$,

$$T(g(\omega)(z)) + (\omega - z)h(\omega)(z) = k(z), \quad z \in \mathbb{A}.$$

Definition

A family $\{F_\theta\}_{\theta\in(0,1)}$ of interpolation functors is said to be stable if for any Banach couples $\vec{A}=(A_0,A_1)$ and $\vec{B}=(B_0,B_1)$ and for every operator $S\colon \vec{A}\to\vec{B}$ such that the restriction S_{θ_*} of S to $F_{\theta_*}(\vec{A})$ is invertible for some $\theta_*\in(0,1)$, there exists $\varepsilon>0$ such that, for any $\theta\in I(\theta_*)=(\theta_*-\varepsilon,\theta_*+\varepsilon)$, we have

(i) $S_\theta\colon F_\theta(\vec{A})\to F_\theta(\vec{B})$ are invertible operators;

(ii) $S_\theta^{-1}\colon F_\theta(\vec{B})\to F_\theta(\vec{A})$ agrees with $S_{\theta_*}^{-1}\colon F_{\theta_*}(\vec{B})\to F_{\theta_*}(\vec{A})$ on $B_0\cap B_1$, i.e., $S_\theta^{-1}y=S_{\theta_*}^{-1}y$ for all $y\in B_0\cap B_1$;

(iii) $\sup_{\theta\in I(\theta_*)}\|S_\theta^{-1}\|_{F_\theta(\vec{B})\to F_\theta(\vec{A})}\leqslant C\|S_{\theta_*}^{-1}\|_{F_{\theta_*}(\vec{B})\to F_{\theta_*}(\vec{A})}$ for some $C=C(\theta_*)$.

Theorem

If $\vec{\mathcal{X}}=(\mathcal{X}_0,\mathcal{X}_1)$ is a Banach couple of translation and rotation invariant pseudolattices, then the following family of interpolation functors $\{F_\theta\}_{\theta\in(0,1)}$ is stable, where

$$F_\theta(A_0,A_1)\cong(A_0,A_1)_{\vec{\mathcal{X}},e^\theta}$$

for any Banach couple (A_0,A_1).

- Remark. Let $\{F_\theta\}_{\theta\in(0,1)}$ be a stable family of interpolation functors and let $T\colon (X_0, X_1) \to (Y_0, Y_1)$. Then the set of all $\theta \in (0,1)$ for which

$$T\colon F_\theta(X_0, X_1) \to F_\theta(Y_0, Y_1)$$

is invertible, is open, so it is a union of open disjoint intervals. These intervals we will call intervals of invertibility of T with respect to the family $\{F_\theta\}_{\theta\in(0,1)}$.

- Question. Let $I \subset (0,1)$ be any interval of invertibility of T. Is it true that for any θ, $\theta' \in I$ the inverses T_θ^{-1} and $T_{\theta'}^{-1}$ agree on

$$F_\theta(Y_0, Y_1) \cap F_{\theta'}(Y_0, Y_1)?$$

Theorem

Let $1 \leqslant q \leqslant \infty$ and let $T\colon (X_0, X_1) \to (Y_0, Y_1)$ and let $I \subset (0,1)$ be an interval of invertibility of T with respect to the family $\{(\cdot)_{\theta,q}\}_{\theta\in(0,1)}$ of real interpolation functors. Then for any θ_0, $\theta_1 \in I$,

$$T_{\theta_0}^{-1}(y) = T_{\theta_1}^{-1}(y), \quad y \in (Y_0, Y_1)_{\theta_0,q} \cap (Y_0, Y_1)_{\theta_1,q}.$$

Theorem

Let $T\colon (X_0, X_1) \to (Y_0, Y_1)$ be an operator between couples of complex Banach spaces and let $I \subset (0,1)$ be an interval of invertibility of T with respect to the family $\{[\,\cdot\,]_\theta\}_{\theta\in(0,1)}$. Then for any θ_0, $\theta_1 \in I$,

$$T_{\theta_0}^{-1}(y) = T_{\theta_1}^{-1}(y), \quad y \in [Y_0, Y_1]_{\theta_0} \cap [Y_0, Y_1]_{\theta_1}.$$

Theorem

Let $T\colon (X_0, X_1) \to (Y_0, Y_1)$ *be an operator between couples of complex Banach spaces. If* $T\colon [X_0, X_1]_{\theta_*} \to [Y_0, Y_1]_{\theta_*}$ *is invertible for some* $\theta_* \in (0,1)$*, then*

$$T\colon (X_0, X_1)_{\theta_*,q} \to (Y_0, Y_1)_{\theta_*,q}$$

is invertible for all $q \in [1, \infty]$*.*

Theorem

If $T\colon (X_0, X_1) \to (Y_0, Y_1)$ *is such that* $T\colon [X_0, X_1]_{\theta_*} \to [Y_0, Y_1]_{\theta_*}$ *is Fredholm then for all* $1 \leqslant q \leqslant \infty$ *the operator*

$$T\colon (X_0, X_1)_{\theta_*,q} \to (Y_0, Y_1)_{\theta_*,q}$$

is Fredholm and index is the same

$$\operatorname{ind}(T|_{(X_0,X_1)_{\theta_*,q}}) = \operatorname{ind}(T|_{[X_0,X_1]_{\theta_*}}).$$

Theorem

Suppose that an operator $T\colon (X_0, X_1) \to (Y_0, Y_1)$ *between Banach spaces is such that the operator*

$$T|_{(X_0,X_1)_{\theta_*,\infty}}\colon (X_0, X_1)_{\theta_*,\infty} \to (Y_0, Y_1)_{\theta_*,\infty}$$

is Fredholm. Then there exists $\varepsilon > 0$ *such that for all* $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$ *the operator*

$$T|_{(X_0,X_1)_{\theta,\infty}}\colon (X_0, X_1)_{\theta,\infty} \to (Y_0, Y_1)_{\theta,\infty}$$

is Fredholm.

Definition. A family of interpolation functors $\{F_\theta\}_{\theta\in(0,1)}$ satisfies the (Δ)-condition if for any Banach couple $\vec{A} = (A_0, A_1)$ and for any θ_0, θ_1 with $0 < \theta_0 < \theta_1 < 1$, we have continuous inclusions

$$F_{\theta_0}(\vec{A}) \cap F_{\theta_1}(\vec{A}) \hookrightarrow \bigcap_{\theta_0<\theta<\theta_1} F_\theta(\vec{A}) \hookrightarrow (F_{\theta_0}(\vec{A}))^c \cap (F_{\theta_1}(\vec{A}))^c,$$

where the norm in $\bigcap_{\theta_0<\theta<\theta_1} F_\theta(\vec{A})$ is given by

$$\|a\|_{\bigcap_{\theta_0<\theta<\theta_1} F_\theta(\vec{A})} = \sup_{\theta_0<\theta<\theta_1} \|a\|_{F_\theta(\vec{A})}.$$

and the Gagliardo completion $(F_{\theta_i}(\vec{A}))^c,\ j \in \{0, 1\}$ is taken with respect to the sum $F_{\theta_0}(\vec{A}) + F_{\theta_1}(\vec{A})$.

Definition. A family of interpolation functors $\{F_\theta\}_{\theta\in(0,1)}$ satisfies the reiteration condition if for any Banach couple $\vec{A} = (A_0, A_1)$ and for any θ_0, θ_1, $\lambda \in (0,1)$, we have

$$F_\lambda(F_{\theta_0}(\vec{A}), F_{\theta_1}(\vec{A})) = F_{(1-\lambda)\theta_0+\lambda\theta_1}(\vec{A}).$$

Theorem

Let $T\colon (X_0, X_1) \to (Y_0, Y_1)$ and let $I \subset (0,1)$ be an interval of invertibility of T with respect to the stable family of interpolation functors $\{F_\theta\}_{\theta\in(0,1)}$. Assume that $\{F_\theta\}_{\theta\in(0,1)}$ satisfy both the (Δ) and the reiteration condition. Then for any $\theta_0, \theta_1 \in I$, the inverse operators $T_{\theta_0}^{-1}$ and $T_{\theta_1}^{-1}$ agree on $F_{\theta_0}(\vec{Y}) \cap F_{\theta_1}(\vec{Y})$, i.e.,

$$T_{\theta_0}^{-1}(y) = T_{\theta_1}^{-1}(y), \quad y \in F_{\theta_0}(\vec{Y}) \cap F_{\theta_1}(\vec{Y}).$$

- Let $T\colon E \to F$ be a linear operator between Banach spaces, then the injection modulus of T is defined by

$$j(T) := \inf_{\|x\|_E=1} \|Tx\|_F = \sup\{\tau > 0;\ \|Tx\|_F \geqslant \tau\|x\|_E\}.$$

An operator T is called an injection if $j(T) > 0$. Clearly that T that an injection can be characterized as a one-to-one operator from E into F with closed range.

- The surjection modulus of T is given by

$$q(T) := \sup\{\tau > 0;\ T(B_E) \supset \tau B_F\}.$$

An operator T is called a surjection if $q(T) > 0$, which is equivalent to $T(E) = F$. If $\|T\| = q(T) = 1$, then T is said to be a metric surjection (i.e., T maps the open unit ball of E onto the open unit ball of F).

- Definition. Let $\mathcal{G}(\vec{X})$ the Banach space of all continuous functions $g\colon \bar{S} \to X_0 + X_1$ that are analytic on the strip S and grow no faster than $C(1+|z|)$ for some $C > 0$. We endow $\mathcal{G}(\vec{X})$ with the norm

$$\|g\|_{\mathcal{G}} := \max_{j=0,1} \left\{ \sup_{s\neq t} \frac{\|g(j+is) - g(j+it)\|_{X_j}}{|s-t|} \right\}.$$

The upper/second complex interpolation space is defined by

$$[\vec{X}]^{\theta} := \{g'(\theta);\, g \in \mathcal{G}\}$$

and equipped with the quotient norm.

Theorem

Let $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ be couples of Banach spaces, and let an operator $T\colon \vec{X} \to \vec{Y}$. Assume that the unit balls of $[\vec{X}]^{\theta_0}$ and $[\vec{X}]^{\theta}$ are closed in $X_0 + X_1$ for some θ_0 and $\theta \in (0,1)$. If d is a conformal map of the strip S onto the unit open disc $\mathbb{D}$, then

$$j\big(T\colon [\vec{X}]^{\theta} \to [\vec{Y}]^{\theta}\big) \geqslant M \max\left\{\frac{j_{\theta_0}(T) - q(\theta,\theta_0)M}{M - q(\theta,\theta_0)j_{\theta_0}(T)},\, 0\right\},$$

where $j_{\theta_0}(T) = j\big(T\colon [\vec{X}]^{\theta_0} \to [\vec{Y}]^{c}_{\theta_0}\big)$, $M = \|T\|_{\vec{X}\to\vec{Y}}$ and and $q\colon S \times S \to [0,1]$ is given by

$$q(\lambda, z) = \left|\frac{d(\lambda) - d(z)}{1 - \overline{d(z)}d(\lambda)}\right|, \quad \lambda, z \in \mathbb{D}.$$

Corollary

Under the assumptions of the above Theorem the conditions $j_{\theta_0}(T) > 0$ *and* $\frac{j_{\theta_0}(T)}{M} > q(\theta, \theta_0)$ *imply that*

$$j\big(T \colon [\vec{X}]^{\theta} \to [\vec{Y}]^{\theta}\big) > 0$$

i.e., $T \colon [\vec{X}]^{\theta} \to [\vec{Y}]^{\theta}$ *is an isomorphic embedding.*

Remark:

If $\theta_0 \in (0,1)$*, then* $q(\theta, \theta_0) \to 0$ *as* $\theta \to \theta_0$*. Thus* $j_{\theta_0}(T) > 0$ *implies that there exists* $\varepsilon > 0$ *such that*

$$j\big(T \colon [\vec{X}]^{\theta} \to [\vec{Y}]^{\theta}\big) > 0$$

for all θ *in* $(0,1)$ *with* $|\theta - \theta_0| < \varepsilon$.

- Example. The map $z \mapsto \operatorname{tg} z$ is a conformal map of the open strip $\{z \in \mathbb{C};\ -\frac{\pi}{4} < \mathrm{Re} z < \frac{\pi}{4}\}$ onto the disc $\mathbb{D}$. Thus φ defined by

$$\varphi(z) = \operatorname{tg}\left(z - \frac{1}{2}\right)\frac{\pi}{2}, \quad z \in S$$

is a conformal map of S onto $\mathbb{D}$ and so q is given by

$$q(\lambda, z) = \left|\frac{\operatorname{tg}(\lambda - \frac{1}{2})\frac{\pi}{2} - \operatorname{tg}(z - \frac{1}{2})\frac{\pi}{2}}{1 - \operatorname{tg}(\lambda - \frac{1}{2})\frac{\pi}{2}\ \operatorname{tg}(\bar{z} - \frac{1}{2})\frac{\pi}{2}}\right|, \quad \lambda, z \in \mathbb{D}.$$

- Corollary. If $j_{1/2}(T) > 0$, then by

$$q(\theta, 1/2) = \left|\operatorname{tg}\left(\theta - \frac{1}{2}\right)\frac{\pi}{2}\right|,$$

it follows that $j\big(T\colon [\vec{X}]^{\theta} \to [\vec{Y}]^{\theta}\big) > 0$ wheneber

$$\left|\theta - \frac{1}{2}\right| < \frac{2}{\pi}\arctan\left(\frac{j_{1/2}(T)}{M}\right).$$

Theorem

Let $\vec{X} = (X_0, X_1)$ *and* $\vec{Y} = (Y_0, Y_1)$ *be complex Banach couples, and let* $T\colon \vec{X} \to \vec{Y}$ *be an operator with* $M' = \|T'\|_{\vec{Y}' \to \vec{X}'}$. *Then for all* θ_0, $\theta \in (0,1)$,

$$q_\theta(T) \geqslant M' \max\left\{\frac{q_{\theta_0}(T) - q(\theta,\theta_0)M'}{M' - q(\theta,\theta_0)q_{\theta_0}(T)},\, 0\right\},$$

where $q_\theta(T) = q\big(T\colon [\vec{X}]^\theta \to [\vec{Y}]^\theta\big)$.

Let (X_0, X_1) be a Banach couple of complex Banach funtion lattices on a σ-finite measure space (Ω, Σ, μ). The Calderón product $X_0^{1-\theta} X_1^{\theta}$ $(0 < \theta < 1)$ is defined to be the space of all $f \in L^0(\mu)$ such that

$$|f| \leqslant \lambda\, |f_0|^{1-\theta} |f_1|^{\theta}\, quad\, \mu - a.e.$$

for some $\lambda > 0$ and $f_j \in X_j$ with $\|f_j\|_{X_j} \leqslant 1$, $j = 0, 1$. The Calderón product is a Banach functin lattice on (Ω, Σ, μ) equipped with the norm

$$\|f\| = \inf\big\{\lambda > 0 \colon\, |f| \leqslant \lambda\, |f_0|^{1-\theta} |f_1|^{\theta},\, f_0 \in B_{X_0},\, x_1 \in B_{X_1}\big\}.$$

Theorem

Let (X_0, X_1), $\vec{Y} = (Y_0, Y_1)$ be couples of Banach lattices with the Fatou property. Assume that $T\colon (X_0, X_1) \to (Y_0, Y_1)$ is such that $T\colon X_0^{1-\theta_*} X_1^{\theta_*} \to Y_0^{1-\theta_*} Y_1^{\theta_*}$ is an invertible operator for some $\theta_* \in (0,1)$. Then there exists $\delta > 0$ such that

$$T\colon X_0^{1-\theta} X_1^{\theta} \to Y_0^{1-\theta} Y_1^{\theta}$$

is an invertible operator whenever $|\theta - \theta_*| < \delta$.

Let $\Omega \subset \mathbb{R}^n$ be a unbounded domain (above the graph of real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$).

Question: For which $X(\partial\Omega)$ the Dirichlet problem for the Laplacian:

$$\Delta u = 0 \quad \text{in } \Omega$$

under the conditions $M(u) \in X(\partial\Omega)$ and $u|_{\partial\Omega} = f \in X(\partial\Omega)$ has a solution? Here, M stands for the nontangential maximal operator given by

$$M(u)(x) := \sup\{|u(y)|;\, y \in \Omega,\, |x-y| < 2\,\mathrm{dist}(y, \partial\Omega)\}, \quad x \in \partial\Omega$$

and the trace is taken in the sense of nontangential convergence to the boundary, i.e.,

$$u|_{\partial\Omega}(x) := \lim_{\substack{\Omega \ni y \to x \\ |x-y| < 2\,\mathrm{dist}(y,\partial\Omega)}} u(y), \quad x \in \partial\Omega.$$

- R. Coifman, A. McIntosh and Y. Meyer (1982); G. Verechota (1984).

$$\|M(\mathcal{D}f)\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{L^p(\partial\Omega)}, \quad \mathcal{D}f|_{\partial\Omega} = \Big(\frac{1}{2}I + \mathcal{K}\Big)f,$$

for every $f \in L^p(\partial\Omega)$, $1 < p < \infty$ and the solution is given by

$$u(x) = \mathcal{D}\Big(\Big(\frac{1}{2}I + \mathcal{K}\Big)^{-1} f\Big)(x), \quad x \in \Omega$$

whenever the inverse $\left(\frac{1}{2}I + \mathcal{K}\right)^{-1}$ exists in $(L^p(\partial\Omega)$. Here $\mathcal{D}$ is the harmonic double layer potential operator and $\mathcal{K}$ its principial-value boundary version.

- B. Dalhberg and C. Kenig (1987) There exists $\varepsilon > 0$ such that

$$\frac{1}{2}I + \mathcal{K}\colon L^p(\partial\Omega) \to L^p(\partial\Omega)$$

isomorphically for $p \in (2 - \varepsilon, \infty)$.