Basic properties of Toeplitz and Hankel operators in non-algebraic setting

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Paweł Domański Memorial Conference, Będlewo 2018

$L^0 := L^0(\mathbb{T}, m)$ - the space of all measurable complex-valued, almost everywhere finite functions on $\mathbb{T}$.

quasi-Banach function space (q-B.f.s.)

A quasi Banach space $X \subset L^0$ such that

- if $f \in X, g \in L^0$ and $|g| \leqslant |f|$ -a.e., then $g \in X$ and $\|g\|_X \leqslant \|f\|_X$,
- $\chi_E \in X$ for each measurable set $E \subset \mathbb{T}$ (i.e. $L^\infty \subset X$).

If, in addition, X is a Banach space, we will use the abbreviation B.f.s. for X.

Köthe dual

For a q-B.f.s. X, its Köthe dual X' is defined as the space of functions $g \in L^0$ satisfying

$$\|g\|_{X'} = \sup\left\{\int_{\mathbb{T}} |f(t)g(t)|\, dm(t) \ : \|f\|_X \leqslant 1\right\} < \infty.$$

Fatou property

A B.f.s. X has the Fatou property iff $X = X''$

Order continuity

$f \in X$ is said to be an order continuous element if for each $(f_n)_{n\in\mathbb{N}} \subset X$, $0 \leqslant f_n \leqslant |f|$ with $f_n \to 0$ a.e., there holds $\|f_n\|_X \to 0$.
The subspace of order continuous elements of X is denoted by X_o.

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The subspace of order continuous elements of X is denoted by X_o.

The distribution function μ_f of $f \in L^0$ is given by

$$\mu_f(\lambda) = m\{t \in \mathbb{T} : |f(t)| > \lambda\}, \quad \lambda \geqslant 0.$$

$f, g \in L^0$ are equimeasurable if

$$\mu_f \equiv \mu_g$$

The non-increasing rearrangement f^* of $f \in L^0$ is defined by

$$f^*(x) = \inf\{\lambda : \mu_f(\lambda) \leqslant x\}, \quad x \geqslant 0.$$

Rearrangement invariant space

A q-B.f.s. X is called rearrangement-invariant (r.i. q-B.f.s. for short) if for every pair of equimeasurable functions $f, g \in L^0$

$$f \in X \Rightarrow g \in X \text{ and } \|f\|_X = \|g\|_X.$$

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Dilation operator

Let X be a r.i. q-B.f. space. For each $s \in \mathbb{R}_+$ the dilation operator D_s is defined as

$$(D_s f)(e^{i\theta}) = \begin{cases} f(e^{i\theta s}), & \theta s \in [0, 2\pi), \\ 0, & \theta s \notin [0, 2\pi), \end{cases} \qquad \theta \in [0, 2\pi).$$

Boyd indices

The limits

$$\alpha_X = \lim_{s \to 0^+} \frac{\log \|D_{1/s}\|_{X \to X}}{\log s}, \quad \beta_X = \lim_{s \to \infty} \frac{\log \|D_{1/s}\|_{X \to X}}{\log s}$$

are called the lower and upper Boyd indices of X, respectively. We say that the Boyd indices are nontrivial if $\alpha_X, \beta_X \in (0, 1)$.

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Pointwise multipliers

Let X and Y be B.f.s. The space of pointwise multipliers $M(X, Y)$ is defined by

$$M(X, Y) = \{f \in L^0 : fg \in Y \text{ for all } g \in X\} \tag{1}$$

with the norm

$$\|f\|_{M(X,Y)} = \sup\{\|fg\|_Y : \ g \in X, \|g\|_X \leqslant 1\}. \tag{2}$$

Each $f \in M(X, Y)$ is the symbol of multiplication operator

$$M_f : g \mapsto fg, \quad M_f : X \to Y.$$

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Examples

- $M\left(E, L^1\right) \equiv E'$ - Köthe dual of E.
- If $1 \leqslant q < p < \infty, 1/r = 1/q - 1/p$, then

$$M(L^p, L^q) \equiv L^r.$$

- Let $1 \leqslant p < q < \infty$, then $M(L^p, L^q) = \{0\}$.
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$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi_2},$$

where

$$\varphi_2\left(u\right) = \sup_{v>0} \left\{\varphi\left(uv\right) - \varphi_1\left(v\right)\right\}.$$

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Pointwise product

For a given two B.f. spaces X and Y we define pointwise product space $X \odot Y$ as

$$X \odot Y = \{xy : x \in X \text{ and } y \in Y\}, \tag{3}$$

with the quasi-norm $\|\cdot\|_{X \odot Y}$ given by the formula

$$\|z\|_{X \odot Y} = \inf \{\|x\|_X \|y\|_Y : z = xy,\ x \in X \text{ and } y \in Y\}. \tag{4}$$

For $n \in \mathbb{Z}$ and $t \in \mathbb{T}$, let $\chi_n(t) := t^n$.
The Fourier coefficients of a function $f \in L^1$ are given by

$$\widehat{f}(n) := \langle f, \chi_n \rangle, \quad n \in \mathbb{Z},$$

where

$$\langle f, g \rangle := \int_{\mathbb{T}} f(t)\overline{g(t)}\, dm(t).$$

Hardy spaces

Let X be a r.i. q-B.f.s. such that $X \subset L^1$. Hardy space $H[X]$ is defined as

$$H[X] := \left\{ f \in X : \ \widehat{f}(n) = 0 \quad \text{for all} \quad n < 0 \right\},$$

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Riesz projection

$P : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ (P is bounded on r.i. Y when it has nontrivial Boyd indices)

$$P : \sum_{n=-\infty}^{\infty} \widehat{f}(n)t^n \mapsto \sum_{k=0}^{\infty} \widehat{f}(n)t^n.$$

Toeplitz operator

$a \in L^\infty$ – algebraic case $a \in M(X, Y)$ – non-algebraic case

$$T_a : f \mapsto PM_af \quad T_a : H^2 \to H^2 \quad T_a : H[X] \to H[Y]$$

Flip operator

$J : L^1 \to L^1$ (J is an isometry on r.i. X)

$$Jf(t) = t^{-1}f(t^{-1})$$

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Toeplitz matrix

$(a_n)_{n=-\infty}^{\infty}$ - sequence of complex numbers

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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$(a_n)_{n=1}^{\infty}$ - sequence of complex numbers

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & a_4 & \dots \\ a_3 & a_4 & a_5 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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General Nehari Theorem

Let X, Y be two r.i. B.f. spaces, such that X is separable, $X \subset Y$, Y has nontrivial Boyd indices and one of the following conditions holds:
i) $X \odot M(X,Y) = Y$ and $X, Y \in (FP)$,
ii) $\beta_X < \alpha_Y$.
If a continuous linear operator $A : H[X] \to H[Y]$ is such that

$$\langle A\chi_j, \chi_k \rangle = a_{k+j+1}$$

for $j, k \geqslant 0$ and some sequence $(a_k)_{k>0}$, then there exists $a \in M(X,Y)$ such that $\hat{a}(n) = a_n$ for $n > 0$ and $A = H_a$, i.e. $A : f \mapsto PaJf$. Moreover,

$$c\,\mathrm{dist}_{M(X,Y)}(a, \overline{H[M(X,Y)]}) \leqslant \|H_a\|_{H[X]\to H[Y]}$$

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Measuring compactness/noncompactness

- Given a set A in a Banach space X, its **Kuratowski measure of noncompactness** $\alpha(A)$ is defined as

$$\alpha(A) = \inf\{\delta > 0 : A \subset \sum_{k=1}^{N} B_k, diam(B_k) \leqslant \delta \text{ and } N < \infty\}.$$

- For a bounded operator $T : X \to Y$, its **Kuratowski measure of noncompactness** $\alpha(T)$ is just the measure of noncompactness of the set $T(B(X))$ in Y, where $B(X)$ is the unit ball of X, i.e.

$$\alpha(T) := \alpha(T(B(X))).$$

- For $T : X \to Y$ the **essential norm** is given by

$$\|T\|_e := \inf\{\|T - K\| : K : X \to Y \text{ is compact}\}$$

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$$\alpha(A) = \inf\{\delta > 0 : A \subset \sum_{k=1}^{N} B_k, diam(B_k) \leqslant \delta \text{ and } N < \infty\}.$$

- For a bounded operator $T : X \to Y$, its **Kuratowski measure of noncompactness** $\alpha(T)$ is just the measure of noncompactness of the set $T(B(X))$ in Y, where $B(X)$ is the unit ball of X, i.e.

$$\alpha(T) := \alpha(T(B(X))).$$

- For $T : X \to Y$ the **essential norm** is given by

$$\|T\|_e := \inf\{\|T - K\| : K : X \to Y \text{ is compact}\}$$

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Noncompactness of Toeplitz operators

Theorem

Let X, Y be r.i. B.f. spaces such that $X \subset Y$ and Y has nontrivial Boyd indices. Suppose $a \in M(X, Y)$. Then the Kuratowski measure of noncompactness of the Toeplitz operator T_a satisfies

$$\alpha(T_a) \geqslant (\gamma\beta)^{-1} \max_{n \in \mathbb{Z}} |\widehat{a}(n)|, \tag{5}$$

for γ and β being constants of inclusions $Y \subset^{\gamma} L^1$ and $L^{\infty} \subset^{\beta} X$.

Compactness of Hankel operators

Hartman Theorem

A Hankel operator $H_a : H^2 \to H^2$ is compact if and only if $a \in \overline{H^\infty} + C$.

Theorem

Let X, Y be r.i. B.f. spaces with the Fatou property such that $X \subset Y$ ($X \neq Y$) and Y has nontrivial Boyd indices. For $a \in M := M(X, Y)$ and the Hankel operator $H_a : H[X] \to H[Y]$ there holds:

(a)
$$\|H_a\|_e \leqslant \|P\|_{Y \to Y} \operatorname{dist}_M(a, M_o + \overline{H[M]}).$$

In particular, if $a \in M_o + \overline{H[M]})$ then H_a is compact.

(b) If spaces X, Y satisfy assumptions of the General Nehari Theorem and X is reflexive, then also

$$\|H_a\|_e \geqslant c \operatorname{dist}_M(a, M_o + \overline{H[M]}),$$

for some constant $c > 0$ depending only on spaces X, Y.

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Corollary

(a) X, Y - r.i. B.f. spaces, $X \subset Y$ and Y has nontrivial Boyd indices. If $M(X, Y)$ is separable, then for each $a \in M(X, Y)$ the Hankel operator $H_a : H[X] \to H[Y]$ is compact.

(b) If $1 < q < p < \infty$ then all Hankel operators from H^p to H^q are compact.

(c) Let $1 < p_2 < p_1 < \infty$.

(c1) If $1 \leqslant q_1 < q_2 \leqslant \infty$ then each Hankel operator $H_a : H^{p_1,q_1} \to H^{p_2,q_2}$ is compact.

(c2) If $1 \leqslant q_2 \leqslant q_1 < \infty$, then a Hankel operator $H_a : H^{p_1,q_1} \to H^{p_2,q_2}$ is compact if and only if $Pa \in H[L_o^{p,\infty}]$, where $\frac{1}{p} = \frac{1}{p_2} - \frac{1}{p_1}$.

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Commutators - does it make sense?

Commutator

If $T, S : X \to X$ then the commutator is defined as

$$[T, S] = TS - ST.$$

Commutator of Toeplitz operators

Let T_a, T_b be two Toeplitz operators.

$$[T_a, T_b] = T_a T_b - T_b T_a.$$

Semi-commutator of Toeplitz operators

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Commutators - example

- Let $1 < q < r < p < \infty$ and $a \in M(L^p, L^r)$, $b \in M(L^r, L^q)$.
- Then $a \in L^{h_0}$ and $b \in L^{h_1}$, where $1/h_0 = 1/r - 1/p$ and $1/h_1 = 1/q - 1/r$.
- Also $a \in M(L^s, L^q) = L^{h_0}$ and $b \in M(L^p, L^s) = L^{h_1}$, where $1/s - 1/p = 1/q - 1/r$.

$$\begin{array}{ccc} H^p & \xrightarrow{T_a} & H^r \\ \downarrow T_b & \overset{T_{ab}}{\searrow} & \downarrow T_b \\ H^s & \xrightarrow{T_a} & H^q \end{array} \tag{6}$$

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Commutators - general situation

$$\begin{array}{ccc} H[X] & \xrightarrow{T_a} & H[Y] \\ \Big\downarrow T_b & & \Big\downarrow T_b \\ H[W=?] & \xrightarrow{T_a} & H[Z] \end{array} \tag{7}$$

Theorem

Let $X \subset Y \subset Z$ be r.i. B.f. spaces. If $M_a : X \to Y$ and $M_b : Y \to Z$ and $W = X \odot M(Y, Z)$ then the following diagram commutes

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Compact commutators

Axler–Chang–Sarason–Volberg

$[T_a, T_b)$ is compact on H^2 if and only if $H^\infty[\bar{a}] \cap H^\infty[b] \subset H^\infty + C$.

Theorem

Let $X \subset Y \subset Z$ be r.i. B.f. spaces such that Y, Z and $W = X \odot M(Y, Z)$ have nontrivial Boyd indices and W is a Banach space. Assume that $a \in M(X, Y), b \in M(Y, Z)$.
If $a \in M(X, Y)_o$ or $b \in M(Y, Z)_o$, then the commutator $[T_a, T_b]$ and the semi-commutator $[T_a, T_b)$ are compact.

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Regularization

Let $f \in Z := X \odot Y$ i.e.

$$f = gh \text{ for some } g \in X, h \in Y \text{ and } \|f\|_Z \approx \|g\|_X \|g\|_Y.$$

Question:

Suppose that f has some additional property (is analytic, smooth, simple, etc.). Can we choose $g \in X$ and $h \in Y$ with the same property in such a way that

$$f = gh \text{ and } \|f\|_Z \approx \|g\|_X \|g\|_Y?$$

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Regularization for step functions

Theorem

Let X and Y be two r.i.B.f. spaces. Suppose that $z \in X \odot Y$ is of the form

$$z = \sum_{n=1}^{\infty} c_n \chi_{A_n},$$

where (A_n) is any sequence of pairwise disjoint sets. Then $\|z\|_{X \odot Y}$ is attained on elements of the same form. In other words,

$$\|z\|_{X \odot Y} = \inf \Big\{ \|x\|_X \|y\|_Y : z = xy,\ x = \sum_{n=1}^{\infty} a_n \chi_{A_n} \in X,\ y = \sum_{n=1}^{\infty} b_n \chi_{A_n} \in Y \Big\}.$$

Regularization for analytic functions

Theorem

Let X and Y be two L-convex q-B.f. spaces and $X \subset Y$. Then:

1. $H[M(X,Y)] = M(H[X], H[Y])$,
2. $M(X,Y) = M(H[X], Y)$.

Moreover, when X, Y are B.f.s., then the above spaces have also equal norms.

Theorem

Let X, Y be two q-B.f. spaces. Then

1. $H[X] \odot H[Y] = H[X \odot Y]$;
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with equality of semi-norms.

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Applications

p-convexity

A B.f.s. X is called p-convex when there is $C > 0$ such that for arbitrary $(x_k)_{k=1}^n \in X$

$$\|(\sum_{k=1}^n |x_k|^p)^{1/p}\|_X \leqslant C(\sum_{k=1}^n \|x_k\|_X^p)^{1/p}.$$

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A B.f.s. X is called p-concave when there is $C > 0$ such that for arbitrary $(x_k)_{k=1}^n \in X$

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Applications

Factorization Theorem

Let X, Y be two r.i. B.f.spaces such that $X \subset Y$ and Y has nontrivial Boyd indices. If X is p-convex and Y is p-concave for some $1 < p < \infty$, then each Toeplitz operator $T_a : H[X] \to H[Y]$ factorizes strongly through H^p, i.e. there are $b \in M(L^p, Y)$ and $\phi \in H[M(X, L^p)]$ such that $a = b\phi$ and $T_a = T_b M_\phi$, i.e. the diagram commutes

$$\begin{array}{ccc} H[X] & \xrightarrow{T_a} & H[Y] \\ \downarrow M_\phi & \nearrow T_b & \\ H^p & & \end{array}$$

Thank you!