# Extension of weighted vector-valued functions

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# Main Question

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- $E$ be an lcHs over $\mathbb{K}$ , $\mathcal{FV}(\Omega, E)$ a space of weighted functions,
- $\Lambda \subset \Omega$ , $H \subset E'$ ,
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- bounded $\mathcal{C}^{\infty}$: Frerick, Jordá, Wengenroth ’09

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- $\omega_m := \Omega$ for $m \in \mathcal{M}_{\mathrm{top}}$ and $\omega_m \neq \varnothing$ for $m \in \mathcal{M}_0 \cup \mathcal{M}_r$
- **Assumption:** $\mathcal{FV}(\Omega, E)$ lcHs and $\delta_x \in \mathcal{FV}(\Omega, E)'$, $x \in \Omega$

**Goal:** $\mathcal{FV}(\Omega)\varepsilon E := L_e\left(\mathcal{FV}(\Omega)'_\kappa, E\right) \overset{\sim}{\hookrightarrow} \mathcal{FV}(\Omega, E)$

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## Definition (consistent)

$\left(T^E_m, T^{\mathbb{K}}_m\right)_{m\in\mathcal{M}}$ a consistent family $:\Leftrightarrow \forall\, u \in \mathcal{FV}(\Omega)\varepsilon E,\, m \in \mathcal{M},\, x \in \omega_m$:

1. $\Psi(u) \in \operatorname{dom} T^E_m$ and $T^{\mathbb{K}}_{m,x} \in \mathcal{FV}(\Omega)'$
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## Lemma

Let

- $\left(T^E_m, T^{\mathbb{K}}_m\right)_{m\in\mathcal{M}}$ be a consistent family.

Then

$$\Psi\colon \mathcal{FV}(\Omega)\varepsilon E \stackrel{\sim}{\hookrightarrow} \mathcal{FV}(\Omega, E).$$

# Extension from thin sets

- $T_m^{\mathbb{K}} \colon \mathbb{K}^{\Omega} \supset \operatorname{dom} T_m^{\mathbb{K}} \to \mathbb{K}^{\omega_m}, \qquad m \in \mathcal{M} = \mathcal{M}_{\mathrm{top}} \cup \mathcal{M}_0 \cup \mathcal{M}_r$

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$U \subset \bigcup_{m \in \mathcal{M}} \{m\} \times \omega_m$ set of uniqueness for $\mathcal{FV}(\Omega)$ if

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$\Rightarrow$ $\operatorname{span}\{T_{m,x}^{\mathbb{K}} \mid (m, x) \in U\}$ is $\sigma(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$-dense

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Set $\mathcal{FV}_H(U, E)$ as the space of functions $f\colon U \to E$ such that

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## Question

Let $(T^E_m, T^{\mathbb{K}}_m)_{m\in\mathcal{M}}$ be consistent. When is the restriction map

$$R_{U,H}\colon \Psi(\mathcal{FV}(\Omega)\varepsilon E) \to \mathcal{FV}_H(U,E),\; f \mapsto (T^E_m(f)(x))_{(m,x)\in U} ,$$

surjective?

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surjective?

- $U \leftrightarrow \{\delta_x \mid x \in \Lambda\}$

# Extension theorem

## Definition (determine boundedness)

Subspace $H \subset E'$ determines boundedness if

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1. $\Psi(\mathcal{FV}(\Omega)\varepsilon E) \subset \mathcal{FV}(\Omega, E)$
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**proof:** Lemma + abstract extension result by Bonet, Frerick, Jordá '07

# Example ($E$ loc complete, $H = E'$)

- The Schwartz space

$$\mathcal{S}(\mathbb{R}^d, E) := \{ f \in \mathcal{C}^\infty(\mathbb{R}^d, E) \mid \forall\, n \in \mathbb{N},\ \alpha \in \mathfrak{A} :\ |f|_{n,\alpha} < \infty \}$$

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- $\mathcal{S}(\mathbb{R}^d, \mathbb{K})$:

$$T_m^{\mathbb{K}} \colon \operatorname{dom} T_m^{\mathbb{K}} \to \mathbb{K}^{\{1\}},\ T_m^{\mathbb{K}}(f)(1) := \int_{\mathbb{R}^d} f(x) h_m(x) \mathrm{d}x\ , \quad m \in \mathbb{N}_0^d\ ,$$

where $h_m$ is the $m$-th Hermite function.

# Example ($E$ loc complete, $H = E'$)

- $\mathcal{S}(\mathbb{R}^d, E)$, $E$ sequentially complete:

$$T_m^E \colon \operatorname{dom} T_m^E \to E^{\{1\}},\ T_m^E(f)(1) := \int_{\mathbb{R}^d} f(x) h_m(x) \mathrm{d}x\ , \quad m \in \mathbb{N}_0^d\ ,$$

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- $\mathcal{S}(\mathbb{R}^d, E)$, $E$ sequentially complete, $U \leftrightarrow \{T^{\mathbb{K}}_{m,1} \mid m \in \mathbb{N}_0^d\}$:

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$$= s(\mathbb{N}_0^d, E)$$

**Thank you for your attention!**

# Space of smooth functions

- $\mathcal{C}^\infty(\Omega, E) := \{f \in W_{\mathcal{M}}(\Omega, E) \mid \forall\, j, n, \alpha : \ |f|_{j,n,\alpha} < \infty\}$ ,
- $W_{\mathcal{M}}(\Omega, E) := \bigcap_{m \in \mathcal{M}} \operatorname{dom} T_m^E$ ,
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where

- $\mathcal{M} := \mathcal{M}_{\text{top}} \cup \mathcal{M}_r$ ,
- $\mathcal{M}_{\text{top}} := \bigcup_{n \in \mathbb{N}_0} \{\beta \mid |\beta| \leq n\}$ and $\mathcal{M}_r := \mathbb{N}_0^d \times S_d$ ,
- $\operatorname{dom} T^E_{(\beta,\sigma)} := \{f \colon \Omega \to E \mid \sigma((\partial^\beta)^E)f \in \mathcal{C}(\Omega, E)\}$ , $T^E_{(\beta,\sigma)} := \sigma((\partial^\beta)^E)f \quad \forall\, (\beta, \sigma) \in \mathbb{N}_0^d \times S_d$
- where $$\sigma((\partial^\beta)^E)f := \frac{\partial^{|\beta|}}{\partial^{\beta_{\sigma(1)}} x_{\sigma(1)} \cdots \partial^{\beta_{\sigma(d)}} x_{\sigma(d)}} f \, .$$

Acknowledgement

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3. $E$ is sequentially complete and for every $f \in \mathcal{FV}(\Omega, E)$ and $f' \in \mathcal{FV}(\Omega)'$ there is a sequence $(f'_k)_{k\in\mathbb{N}}$ in $\mathcal{F}(\Omega)'$ converging to $f'$ in $\mathcal{F}(\Omega)'_\kappa$ such that $R_f^t(f'_k) \in \mathcal{J}(E)$ for every $k \in \mathbb{N}$.

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4. $$\forall\, f \in \mathcal{FV}(\Omega, E),\, j \in J,\, n \in N\, \exists\, K \in \tau(E):\ N_{j,n}(f) \subset K$$

   where

   $$N_{j,n}(f) := \{T_m^E(f)(x)\nu_{j,n,m}(x) \mid x \in \Omega,\ m \in M_n\}\,.$$

is fulfilled and $\mathcal{FV}(\Omega, E) \subset \mathcal{FV}(\Omega, E)_\kappa$, then $\Psi$ is surjective. consistency

strong

Let

$$R_f\colon E' \to \mathcal{FV}(\Omega),\; R_f(e') := e' \circ f\,,$$

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3. $E$ is a semi-Montel or Schwartz space.
4. There is a family $\mathfrak{K}$ of sets and a map $\pi\colon \Omega \times \mathcal{M}_{\mathrm{top}} \to X$ such that $\bigcup_{K\in\mathfrak{K}} K \subset X$ and the functions of $\mathcal{FV}(\Omega, E)$ vanish at infinity in the weighted topology with respect to $(\pi, \mathfrak{K})$, i.e. every $f \in \mathcal{FV}(\Omega, E)$ fulfils: $\forall\, \varepsilon > 0,\, j \in J,\, n \in N,\, \alpha \in \mathfrak{A}\ \exists\, K \in \mathfrak{K}$:

$$(i)\quad \sup_{\substack{x\in\Omega,\, m\in M_n\\ \pi(x,m)\notin K}} p_\alpha\big(T_m^E(f)(x)\big)\nu_{j,n,m}(x) < \varepsilon$$

$$(ii)\ N_{\pi\subset K,j,n}(f) := \{T_m^E(f)(x)\nu_{j,n,m}(x) \mid (x,m) \in \pi^{-1}(K)\} \in \gamma(E)$$

# Injectivity of the restriction map $R_{U,H}$

surj rest surj psi

## Question

Let $(T^{E}_{m}, T^{\mathbb{K}}_{m})_{m\in\mathcal{M}}$ be consistent. When is the restriction map

$$R_{U,H}\colon \Psi(\mathcal{FV}(\Omega)\varepsilon E) \to \mathcal{FV}_{H}(U,E),\ f \mapsto (T^{E}_{m}(f)(x))_{(m,x)\in U}\ ,$$

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## Proposition

If $(T^E_m, T^{\mathbb{K}}_m)_{m\in\mathcal{M}}$ is strong and consistent, then $R_{U,H}$ is injective.

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1. $\Psi(\mathcal{FV}(\Omega)\varepsilon E) \subset \mathcal{FV}(\Omega, E)$
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**proof:** Let $f \in \mathcal{FV}_H(U, E)$. Choose $X := \operatorname{span}\{T^{\mathbb{K}}_{m,x} \mid (m, x) \in U\}$ and $Y := \mathcal{FV}(\Omega)$. Define $\mathrm{A}\colon X \to E$, $\mathrm{A}(T^{\mathbb{K}}_{m,x}) := f(m, x)$.

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**proof:** Then $H \subset (\mathrm{A}^t)^{-1}(Y)$, use the prop. , set $F(x) := \widehat{\mathrm{A}}(\delta_x)$, $x \in \Omega$.

◂ extension theorem

# Pettis integral

## Definition (Pettis integrable)

Let

- $(\Omega, \Sigma, \mu)$ be a measure space and $E$ an lcs,
- $f\colon \Omega \to E$ such that $e' \circ f \in \mathcal{L}^1(\Omega, \mu)$ for all $e' \in E'$ .

$f$ is called Pettis integrable on $\Omega$ if

$$\exists\ e_\Omega \in E\ \forall\ e' \in E' : \langle e', e_\Omega \rangle = \int_\Omega \langle e', f(x) \rangle \mathrm{d}\mu(x) \ .$$

In this case $e_\Omega$ is unique due to $E$ being Hausdorff and we set

$$\int_\Omega f(x) \mathrm{d}\mu(x) := e_\Omega \ .$$

convolution

# Hermite functions

- For $n \in \mathbb{N}_0$ set

$$h_n\colon \mathbb{R} \to \mathbb{R},\ h_n(x) := (2^n n! \sqrt{\pi})^{-1/2} (x - \frac{d}{dx})^n e^{-x^2/2} \ .$$

- For $n \in \mathbb{N}_0^d$ define the $n$-th Hermite function by

$$h_n\colon \mathbb{R}^d \to \mathbb{R},\ h_n(x) := \prod_{k=1}^{d} h_{n_k}(x_k) \ .$$

example

# Examples ($E$ loc complete, $H = E'$)

- The space $\mathcal{E}\left([-1,1]^d, E\right)$ of all $f \in \mathcal{C}^\infty((-1,1)^d, E)$ such that all partial derivatives of any order can be continuously extended to $[-1,1]^d$ equipped with the system of seminorms given by

$$|f|_{n,\alpha} := \sup_{\substack{x\in(-1,1)^d \\ \beta\in\mathbb{N}_0^d, |\beta|\leq n}} p_\alpha\left((\partial^\beta)^E f(x)\right) .$$

$$U \leftrightarrow \{\delta_x \mid x \in (-1,1)^d \cap \mathbb{Q}^d\}$$

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- $L_c(\Omega, E)$ where $\Omega$ is (DFS), $E$ quasi-complete, $U \leftrightarrow \{\delta_x \mid x \in X\}$, $X \subset \Omega$ dense

◂ example