Symmetrization of some quasi-Banach function spaces

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PAWEŁ DOMAŃSKI MEMORIAL CONFERENCE

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The outline of the talk

1. Introduction.
2. The commutativity property of symmetrization with some known constructions.
3. Some factorization results.

The talk is supported by the Ministry of Science and Higher Education of Poland, grant number 04/43/DSPB/0094 and it is based on the papers:

1. Paweł Kolwicz, Karol Leśnik and Lech Maligranda, Pointwise products of some Banach function spaces and factorization, J. Funct. Anal. 266, 2, (2014), 616–659.
2. P. Kolwicz, K. Leśnik and L. Maligranda, Symmetrization, factorization and arithmetic of quasi-Banach function spaces, submitted, avalaible on https://arxiv.org/pdf/1801.05799.pdf.

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1. if $x \in E, y \in L^0$ and $|y| \leq |x|$ μ-a.e., then $y \in E$ and $\|y\|_E \leq \|x\|_E$;
2. there exists a function x in E that is positive on the whole I.

Introduction.

- Symmetric function space

  By a symmetric quasi-Banach function space on I, where $I=(0,1)$ or $I=(0,\infty)$ with the Lebesgue measure m, we mean a quasi-Banach function space $E=(E,\|\cdot\|_E)$ with the additional property that for any two equimeasurable functions $x\sim y$, $x,y\in L^0(I)$ (that is, $d_x=d_y$, where

$$d_x(\lambda)=m(\{t\in I:|x(t)|>\lambda\}),\lambda\geq 0)$$

  and $x\in E$ we have $y\in E$ and $\|x\|_E=\|y\|_E$. In particular, $\|x\|_E=\|x^*\|_E$, where

$$x^*(t)=\inf\{\lambda>0:\ d_x(\lambda)\leq t\},\ \ t\geq 0.$$

The space of pointwise multipliers.

- Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be quasi-Banach function spaces. The space of pointwise multipliers $M(E, F)$ is defined by

$$M(E,F) = \{x \in L^0(I) : xy \in F \text{ for all } y \in E\} \tag{1}$$

and the functional on it

$$\|x\|_{M(E,F)} = \sup\{\|xy\|_F,\ y \in E, \|y\|_E \leq 1\} \tag{2}$$

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- Note that $M(E,F)$ can be $\{0\}$.
- It is possible that *supp* $M(E,F)$ is smaller than *supp* $E \cap$ *supp* F.

Pointwise products.

- Given two quasi-Banach function spaces E and F define the *pointwise product space* $E \odot F$ as

$$E \odot F = \{x \cdot y : x \in E \text{ and } y \in F\} . \tag{3}$$

with a functional $\|\cdot\|_{E \odot F}$ defined by the formula

$$\|z\|_{E \odot F} = \inf \{\|x\|_E \, \|y\|_F : z = xy, x \in E, y \in F\} . \tag{4}$$

The Calderón-Lozanovskiĭ space (construction).

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$$\rho(E,F)=\{x\in L^0(I):$$

$$|x|\le\lambda\,\rho(|x_0|,|x_1|) \text{ a.e. on } I \tag{5}$$

for some $x_0\in E, x_1\in F$ with $\|x_0\|_E\le 1$, $\|x_1\|_F\le 1$ and for some $\lambda>0\}$.

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- The quasi-norm

$$\|x\|_\rho = \|x\|_{\rho(E,F)} = \inf\{\lambda > 0\}$$

for which the above inequality holds.

The Calderón-Lozanovskiĭ space (construction)

Examples

- If $\rho(u, v) = u^{\theta} v^{1-\theta}$ with $0 < \theta < 1$ we write $E^{\theta} F^{1-\theta}$ instead of $\rho(E, F)$ and these are Calderón spaces.

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- For $1 < p < \infty$, a *p-convexification* $E^{(p)}$ of E is a special case of Calderón space

$$\begin{aligned} E^{1/p}(L^\infty)^{1-1/p} &= E^{(p)} = \{x \in L^0 \colon |x|^p \in E\} \\ \text{and } \|x\|_{E^{(p)}} &= \||x|^p\|_E^{1/p}. \end{aligned}$$

Pointwise products.

Useful characterization.

- Theorem (KLM 2014). Let E and F be a couple of quasi-Banach function spaces. Then

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- Corollary. Let E and F be a couple of quasi-Banach function spaces. Then $E \odot F$ is a quasi-Banach function space and the triangle inequality is satisfied with constant 2, i.e.,

$$\|x+y\|_{E\odot F} \leq 2\left(\|x\|_{E\odot F} + \|y\|_{E\odot F}\right).$$

The symmetrizations.

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- Examples.
  1. The Marcinkiewicz spaces $M_w^{(*)} = (L^\infty(w))^{(*)}$.
  2. The Lorentz spaces $\Lambda_{p,w^p} = (L^p(w))^{(*)}$.
  3. The Lorentz spaces $L^{p,q} = (L^q(w))^{(*)}$ with $w(t) = t^{1/p-1/q}$.
  4. The generalized Orlicz-Lorentz spaces $\Lambda^\varphi = (L^\varphi)^{(*)}$.

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- The *dilation operator* D_s, $s > 0$, $D_s x(t) = x(t/s)\chi_I(t/s)$, $t \in I$, is bounded in any symmetric space E on I and $\|D_s\|_{E\to E} \le \max(1,s)$. A. Kamińska and Y. Raynaud showed that $\|\cdot\|_{E^{(*)}}$ is a quasi-norm if and only if there is a constant $C > 0$ such that
$$\|D_2 x^*\|_E \le C\|x^*\|_E \text{ for all } x^* \in E, \qquad (7)$$

The Hardy operators.

- Consider the Hardy operator H and its formal Köthe dual H^* defined for $x \in L^0(I)$ by

$$Hx(t) = \frac{1}{t}\int_0^t x(s)\,ds, \; H^*x(t) = \int_t^l \frac{x(s)}{s}\,ds \qquad (8)$$

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with $l = m(I), \; t \in I$.

- Note that if $0 < p < 1$, then neither H nor H^* are bounded on $L^p(w)$ spaces for any weight w, therefore we need to consider their "r-convexifications" for $0 < r < \infty$, which are defined by

$$H_r x = [H(|x|^r)]^{1/r} \;\; \text{and} \;\; H_r^* x = [H^*(|x|^r)]^{1/r}, \tag{9}$$

provided the corresponding integrals are finite.

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Theorem

(A. Kamińska and Y. Raynaud 2009, KLM 2018) Let E be a quasi-normed ideal space on I. The following statements are equivalent:
(i) $E^{(*)}$ is a linear space.
(ii) For each $x \in E^{\downarrow}$ we have $D_2 x \in E^{\downarrow}$.
(iii) There is a constant $1 \leq A < \infty$ such that $\|D_2 x\|_E \leq A \|x\|_E$ for all $x \in E^{\downarrow}$.
(iv) $(E^{(*)}, \|\cdot\|_{E^{(*)}})$ is a quasi-normed space.

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- Remark. Let E be a quasi-Banach function space on I and $E^{(*)} \neq \{0\}$. If $\|H_r x\|_E \leq C \|x\|_E$ for all $x \in E^{\downarrow}$, then $\|D_2 x\|_E \leq 2^{1/r} C \|x\|_E$ for all $x \in E^{\downarrow}$.

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- (i) If the operator D_2 is bounded both on $E^{\downarrow}$ and $F^{\downarrow}$ with the constants A_E, A_F, then Calderón–Lozanovskiĭ construction $\rho(E^{(*)}, F^{(*)}) \neq \{0\}$, $\rho(E^{(*)}, F^{(*)}) \subset \rho(E, F)^{(*)}$ and

$$\|x\|_{\rho(E,F)^{(*)}} \leq C_1 \|x\|_{\rho(E^{(*)},F^{(*)})} \quad \text{for all } x \in \rho(E^{(*)}, F^{(*)}) \tag{10}$$

with $C_1 \leq \max(A_E, A_F)$.

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  with $C_1 \leq \max(A_E, A_F)$.
- (ii) If, additionally, the operator H_r^* is bounded on the spaces E, F for some $r > 0$, then $\rho(E, F)^{(*)} \subset \rho(E^{(*)}, F^{(*)})$ and

$$\|x\|_{\rho(E^{(*)},F^{(*)})} \leq C_2 \|x\|_{\rho(E,F)^{(*)}} \quad \text{for all } x \in \rho(E, F)^{(*)} \tag{11}$$

  with $C_2 \leq 2^{1/r} \max(1, 2^{1/r-1}) \cdot \max\left(A_E \|H_r^*\|_{E\to E}, A_F \|H_r^*\|_{F\to F}\right)$.

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- (*iii*) In particular, the inequalities (10) and (11) imply that the functional

$$\|\cdot\|_{\rho(E,F)^{(*)}} \quad (12)$$

is a quasi-norm on the space

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- $\|\cdot\|_{\rho(E,F)^{(*)}}$ is a quasi-norm$\Leftrightarrow$ D_2 is bounded on $\rho(E,F)^{\downarrow}$.

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- $\|\cdot\|_{\rho(E,F)^{(*)}}$ is a quasi-norm $\Leftrightarrow$ D_2 is bounded on $\rho(E,F)^{\downarrow}$.
- Remark. The assumption of theorem (ii) is essential.

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$$\|x\|_{(E \odot F)^{(*)}} \leq (C_1)^2 \|x\|_{E^{(*)} \odot F^{(*)}} \quad \text{for all } x \in E^{(*)} \odot F^{(*)}, \tag{14}$$

where C_1 is the constant from previous theorem with the function $\rho(s,t) = s^{1/2} t^{1/2}$.

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- (ii) If, additionally, the operator H_r^* is bounded on the spaces E, F for some $r > 0$, then $(E \odot F)^{(*)} \subset E^{(*)} \odot F^{(*)}$ and

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  (i) The operator H^* is bounded on the spaces F and $E \odot F'$.
  (ii) For some $r > 0$, the operator H_r^* is bounded on E, F', $\|H_r x^*\|_E \leq C_E \|x^*\|_E$ for all $x^* \in E$ and $\|H_r x^*\|_{F'} \leq C_{F'} \|x^*\|_{F'}$ for all $x^* \in F'$.
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  Then
  $$M(E^{(*)}, F^{(*)}) = M(E, F)^{(*)}.$$
- Apllications for weighted spaces $E = L^p\left(t^a\right), F = L^q\left(t^b\right)$ with $a, b \in \mathbb{R}$ and $1 \leq p, q \leq \infty$.

- For $0 < p, q \leq \infty$ consider the classical *Lorentz function spaces* $L^{p,q} = L^{p,q}(I) = (L^q(w))^{(*)}$ with $w(t) = t^{1/p-1/q}$ on $I = (0,1)$ or $I = (0,\infty)$ defined by the quasi-norms

$$\|x\|_{p,q} = \begin{cases} \left(\int\limits_0^{m(I)} [t^{1/p} x^*(t)]^q \frac{dt}{t} \right)^{1/q}, & \text{for } 0 < p \leq \infty, 0 < q < \infty, \\ \sup\limits_{0<t<m(I)} t^{1/p} x^*(t), & \text{for } 0 < p \leq \infty, q = \infty. \end{cases}$$

Multipliers between Lorentz spaces.

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- $L^{p,p} \equiv L^p$ for $0 < p \leq \infty$ and $L^{\infty,q} = \{0\}$ for $0 < q < \infty$.

Multipliers between Lorentz spaces.

Theorem (KLM 2018). Let $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$ and $I = (0,1)$ or $I = (0,\infty)$.
(i) If either $p_1 < p_2$ or $p_1 = p_2$ and $q_1 > q_2$, then

$$M(L^{p_1,q_1}, L^{p_2,q_2}) = \{0\}.$$

(ii) If either $p_1 > p_2$ or $p_1 = p_2$ and $q_1 \leq q_2$, then

$$M(L^{p_1,q_1}, L^{p_2,q_2}) = L^{p_3,q_3},$$

where

$$\frac{1}{p_3} = \frac{1}{p_2} - \frac{1}{p_1} \text{ and } \frac{1}{q_3} = \begin{cases} \frac{1}{q_2} - \frac{1}{q_1} & \text{if } \quad q_1 > q_2, \\ 0 & \text{if } \quad q_1 \leq q_2 \end{cases} \tag{18}$$

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- **Corollary (KLM 2018).** Suppose the following assumptions are satisfied:
  -from theorem $M(X, Y)^{(*)} = M\left(X^{(*)}, Y^{(*)}\right)$, for the spaces E, F,
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  If F factorizes through E, i.e.,$F = E \odot M(E, F)$, then the symmetrization $F^{(*)}$ factorizes through the symmetrization $E^{(*)}$, that is,
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- **Corollary (KLM 2018).** The factorization of L^p spaces $\longrightarrow$The factorization of $L^p(w)$ spaces$\longrightarrow$The factorization of Lorentz and Marcinkiewicz spaces.

Thank You very much for Your attention