Order continuity in abstract Cesáro function spaces

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Contents

[KK17] T. Kiwerski and P. Kolwicz, *Isomorphic copies of ℓ^∞ in Cesàro-Orlicz function spaces*, Positivity 21, no. 3, 2017, 1015-1030, doi: 10.1007/s11117-016-0449-6,

[KT17] T. Kiwerski, J. Tomaszewski, *Local approach to order continuity in Cesáro function spaces*, J. Math. Anal. Appl. 445, no. 2 2017, 1636-1654, doi: 10.1016/j.jmaa.2017.06.061.

Contents

- In 1968, the Dutch Mathematical Society posted the problem to finding a reprezentation of a dual spaces in the sense of Köthe of Cesáro sequence spaces ces_p and Cesáro function spaces $Ces_p[0,\infty)$, [Pr68]. [1]

[1] [Pr68] Programma van Jaarlijkse Prijsvragen (Annual Problem Section), Nieuw Arch. Wiskd. 16 (1968), 47-51

[2] [KKL48] B. I. Korenblyum, S. G. Kreĭn and B. Ya. Levin, *On certain nonlinear questions of the theory of singular integrals*, 1948

[3] [Le71] G. M. Leibowitz, *A note on the Cesàro sequence spaces*, 1971

[4] [Ja74] A. A. Jagers, *A note on Cesàro sequence spaces*, 1974

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- The space $Ces_\infty[0, 1]$ appeared already in 1948 and it is known as the Korenblyum-Kreĭn-Levin space K, [KKL48]. [2]

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- The space $Ces_\infty[0, 1]$ appeared already in 1948 and it is known as the Korenblyum-Kreĭn-Levin space K, [KKL48]. [2]
- For the first time, the properties of ces_p were studied by Shiue in 1970. In early '70, Leibowitz and Jagers, showed, among others, that ces_p are separable and reflexive spces for $1 < p < \infty$ and $ces_1 = \{0\}$, [Le71] & [Ja74]. [3] [4]

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- Considerations on spaces $Ces_p[0,\infty)$ for $1 \leqslant p \leqslant \infty$ were also initiated by Shiue in 1970, [Sh70]. Later, these spaces were studied by Hassard and Hussein [HH73] Sy, Zhang and Lee [SZL87]. [5] [6] [7]

[5] [Sh70] J. S. Shiue, *A note on Cesàro function space*, 1970
[6] [HH73] B. D. Hassard, D. A. Hussein, *On Cesàro function spaces*, 1973
[7] [SZL87] P. W. Sy, W. Y. Zhang, P. Y. Lee, *The dual of Cesàro function spaces*, 1987
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- Cesáro-Orlicz sequence spaces ces_φ appeared for the first time in Lim and Lee paper from 1988, [LL88]. [8]

[5] [Sh70] J. S. Shiue, *A note on Cesàro function space*, 1970
[6] [HH73] B. D. Hassard, D. A. Hussein, *On Cesàro function spaces*, 1973
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- General considerations of abstract Cesáro spaces CX began to be studied in papers by Leśnik and Maligranda, e.g. [LM15a] & [LM15b]. 9 10

[9] [LM15a] K. Leśnik and M. Maligranda, *Abstract Cesàro Spaces. Duality*, 2015

[10] [LM15b] K. Leśnik and M. Maligranda, *Abstract Cesàro Spaces. Optimal Range*, 2015

[11] [AM14] S. V. Astashkin and L. Maligranda, *Structure of Cesàro function spaces: a survey*, 2014

[12] [Be96] G. Bennett, *Factorizing the Classical Inequalities*, 1996

[13] [LWL96] Y. Q. Liu, B. E. Wu, P. Y. Lee, *Methods of Sequence Spaces*, 1996

- General considerations of abstract Cesáro spaces CX began to be studied in papers by Leśnik and Maligranda, e.g. [LM15a] & [LM15b]. [9] [10]
- For a long time (*for technical reasons*) the structure of Cesáro function spaces have not attracted a lot of attention in contrast to their sequence counterparts (e.g. S. Chen, Y. Cui, H. Hudzik, B. Sims, A. Kamińska, L. Jie, Y. Lie, R. Płuciennik, C. Meng, D. Kubiak, P. Y. Lee, L. Maligranda, N. Petrot, S. Suantai, A. Szymaszkiewicz, cf. references in [AM14] and results in [Be96] & [LWL96]). [11] [12] [13]

[9] [LM15a] K. Leśnik and M. Maligranda, *Abstract Cesàro Spaces. Duality*, 2015

[10] [LM15b] K. Leśnik and M. Maligranda, *Abstract Cesàro Spaces. Optimal Range*, 2015

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- Recently, both isomorphic and isometric structure of Cesáro function spaces attracts the attention of many authors, e.g. Astahskin, Leśnik and Maligranda [ALM15], Curbera and Ricker [CR16], Delgado and Soria [DS07], Kamińska and Kubiak [KK12]. [14] [15] [16] [17]

[14] [ALM15] S. V. Astashkin, K. Leśnik, L. Maligranda, *Isomorphic structure of Cesáro and Tandori spaces*, 2017

[15] [DS08] O. Delgado and J. Soria, *Optimal domain for the Hardy operator*, 2007

[16] [CR16] G. P. Curbera, W. J. Ricker, *Abstract Cesàro spaces: integral representation*, 2016

[17] [KK12] A. Kaminska, D. Kubiak, *On the dual of Cesàro function space*, 2012

Contents

By $L^0 = L^0(I)$ we denote the set of all equivalence classes of real-valued Lebesgue measurable functions defined on $I = [0, 1]$ or $I = [0, \infty)$.

Support of function $f \in L^0(I)$ is defined as

$$\mathrm{supp}(f) := \{t \in I : f(t) \neq 0\}. \quad (1)$$

Banach function space

A Banach space $X := (X, \|\cdot\|)$ is said to be a Banach function space (function space, for short) on I (we write $X[0,1]$ or $X[0,\infty)$) if

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(ii) X satisfies the so-called ideal property, i.e. if $g \in X$, $f \in L^0(I)$ and $|f| \leqslant |g|$ almost everywhere on I, then $f \in X$ and $\|f\| \leqslant \|g\|$.

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(such an element is called weak unit).
In addition, we say that X is non-trivial if $X \neq \{0\}$.
[a]

[a] [BS88] C. Bennet, R. Sharpley, *Interpolation of Operators*, 1988

For two Banach spaces E and F on I the symbol

$$E \hookrightarrow F, \tag{3}$$

means that the embedding $E \subset F$ is continuous, i.e. there is a constant $0 < M < \infty$ (we call it a embedding constant) such that

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Recall that for two Banach function spaces E and F embedding $E \subset F$ is always continuous.
Moreover, $E = F$ means that the spaces are the same as the sets and the norms are equivalent.

For function $f \in L^0(I)$ we define distribution function of f as

$$d_f(\lambda) := m(\{t \in I : |f(t)| > \lambda\}), \tag{5}$$

for $0 \leqslant \lambda \in I$. We say, that two functions $f, g \in L^0$ are equimeasurable when they have the same distribution functions, i.e.

$$d_f \equiv d_g. \tag{6}$$

Symmetric space

By symmetric Banach function space (symmetric space, for short) on I we mean a Banach function space $E = (E, \|\cdot\|_E)$ on I equipped with additional property - for any two equimeasurable functions $f, g \in L^0(I)$

$$\text{if } f \in E, \text{ then } g \in E \text{ and } \|f\|_E = \|g\|_E. \tag{7}$$

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$$\text{if } f \in E, \text{ then } g \in E \text{ and } \|f\|_E = \|g\|_E. \tag{7}$$

In particular, $\|f\|_E = \|f^*\|_E$, where

$$f^*(t) := \inf\{\lambda > 0 : d_f(\lambda) \leqslant t\}, \tag{8}$$

for $t \geqslant 0$ (f^* is called non-increasing rearrangament of function f).

Fundamental function

For a symmetric space E on I its fundamental function ϕ_E is defined by the formula

$$\phi_E(t) := \left\| \chi_{[0,t]} \right\|_E , \tag{9}$$

for $t \in I$.

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for $t \in I$.

Writing $\phi_E(0^+)$ or $\phi_E(\infty)$ we understand $\lim_{t \to 0^+} \phi_E(t)$ or $\lim_{t \to \infty} \phi_E(t)$, respectively.

Order continuity

A point $f \in X$ is said to have an order continuous norm (f is an OC-point) if for any sequence $(f_n) \subset X$ such that

$$0 \leqslant f_n \leqslant |f| \quad \text{and} \quad f_n \to 0 \quad \text{almost everywhere on } I \tag{10}$$

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we have $\|f_n\|_X \to 0$. By X_a we denote the subspace of all order continuous elements of X. Banch function space X is called order continuous (we write $X \in (OC)$) if every element of X has an order continuous norm, i.e.

$$X \in (OC) \iff X_a = X. \tag{11}$$

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The subspace X_a is always closed, see [BS88, Th. 3.8, str. 16].

Note, that $(L^p)_a = L^p$ if $1 \leqslant p < \infty$ ($L^1 \in (OC)$ - *Lebesgue dominated convergence theorem*) and $(L^\infty)_a = \{0\}$ but $(\ell^\infty)_a = c_0$.

- [CKP14, Lemma 2.5 and 2.6] Let E be a symmetric space. Then

$$f \in E_a \iff f^* \in E_a, \quad (12)$$[18]

and

$$f \in E_a \Rightarrow f^*(\infty) = 0. \quad (13)$$

[18] [CKP14] M. Ciesielski, P. Kolwicz, A. Panfil, *Local monotonicity structure of symmetric spaces with application*, 2014

[19] [Lo69] G. Ya. Lozanovskiĭ, *On isomorphic Banach structures*, 1969

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- Let X be a Banach function space. $X \in (OC) \iff X$ contains no isomorphic copy of ℓ^∞. [19]

[18] [CKP14] M. Ciesielski, P. Kolwicz, A. Panfil, *Local monotonicity structure of symmetric spaces with application*, 2014

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- $X_a \subset X_b$, see [BS88, Th. 3.11, p. 18]. Moreover,

$$\{0\} \subsetneq X_a \subsetneq X_b \subsetneq X, \tag{14}$$

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cf. [BS88, Ex. 3, str. 30].

- If X i a symmetric space, then

$$X_a = \{0\} \text{ or } X_a = X_b. \tag{15}$$

Theorem [KT17, Theorem B]

Let E be a symmetric space. The following conditions are equivalent:

(i) $E_a = \{0\}$,

(ii) $E \hookrightarrow L^\infty$,

(iii) $E_a \neq E_b$,

(iv) $\phi_E(0^+) > 0$.

In particular, if $I = [0, 1]$ then condition (ii) is equivalent to the statement that $E = L^\infty$.

For $s > 0$ the dilation operator D_s is defined, on $L^0(I)$, by

$$D_s f(t) := f(t/s), \tag{16}$$

for $t \in I = [0, \infty)$ and by

$$D_s f(t) := \begin{cases} f(t/s) & \text{if } t \leqslant \min\{1, s\}, \\ 0 & \text{if } s < t \leqslant 1, \end{cases} \tag{17}$$

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for $t \in I = [0, 1]$.
Operator D_s is bounded in any symmetric space E and

$$\|D_s\|_{E \to E} \leqslant \max\{1, s\}, \tag{18}$$

see [BS88, str. 148].

$E = L^p$, then $\|D_s\|_{L^p \to L^p} = s^{1/p}$.

Boyd indices

Lower (upper) Boyd indices of symmetric space E are defined by

$$p(E) := \lim_{s\to\infty} \frac{\ln s}{\ln \|D_s\|_{E\to E}}, \tag{19}$$

$$q(E) := \lim_{s\to 0^+} \frac{\ln s}{\ln \|D_s\|_{E\to E}}. \tag{20}$$

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They satisfy the inequalities

$$1 \leqslant p(E) \leqslant q(E) \leqslant \infty.^{a} \tag{21}$$

[a] [Bo69] D. W. Boyd, *Indices of function spaces and their relationship to interpolation*, 1969

Let $1 \leqslant p \leqslant \infty$, then $p(L^p) = q(L^p) = p$.

Theorem [BS88, Th. 6.6 and Th. 6.7, pp. 77-78]

Let $E(I)$ be a symmetric space. Then

$$L^{\infty} \hookrightarrow E[0,1] \hookrightarrow L^{1} \quad \text{and} \quad L^{1} \cap L^{\infty} \hookrightarrow E[0,\infty) \hookrightarrow L^{1} + L^{\infty}. \tag{22}$$

Theorem [BS88, Th. 6.6 and Th. 6.7, pp. 77-78]

Let $E(I)$ be a symmetric space. Then

$$L^\infty \hookrightarrow E[0,1] \hookrightarrow L^1 \quad \text{and} \quad L^1 \cap L^\infty \hookrightarrow E[0,\infty) \hookrightarrow L^1 + L^\infty. \tag{22}$$

Theorem [LT79, Prop. 2.b.3, p. 132]

Let E be a symmetric space. Then for every $1 \leqslant p < p(E)$ and $q(E) < q \leqslant \infty$ we have

$$L^p \cap L^q \hookrightarrow E \hookrightarrow L^p + L^q. \tag{23}$$

In particular, if we take $q = \infty$, then for every $1 \leqslant p < p(E)$

$$L^p \cap L^\infty \hookrightarrow E \hookrightarrow L^p + L^\infty.^a \tag{24}$$

[a] [LT79] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces II. Function Spaces*, 1979

Contents

Abstract Cesáro spaces

For a Banach function space X on I we define an abstract Cesáro space CX by

$$CX = CX(I) := \{f \in L^0(I) : C|f| \in X\}, \tag{25}$$

with the norm $\|f\|_{CX} = \|C|f|\|_X$, where C denote the Cesáro operator

$$C : f \mapsto Cf(x) := \frac{1}{x}\int_0^x f(t)\mathrm{d}t, \tag{26}$$

for $0 < x \in I$.

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- $X = L^p$ and $1 < p \leqslant \infty \Rightarrow Ces_p := CL^p$,
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- $Ces_1[0,\infty) = \{0\}$ and $Ces_1[0,1] = L^1(\ln(\frac{1}{t}))$,
- $X = L^\varphi \Rightarrow Ces_\varphi := CL^\varphi$.

Boyd's Theorem, 1967

Let X be a symmetric space. Then the Cesáro operator $C : X \to X$ is bounded $\iff$ $p(X) > 1$ [a]

[a] [Bo67] D. W. Boyd, *Hilbert transform on rearrangement-invariant spaces, 1967.*

Theorem [KT17, Lemma 2]

Let X be a symmetric space. If $I = [0, 1]$, then $CX \neq \{0\}$. If instead $I = [0, \infty)$, then the following conditions are equivalent:

(i) $CX \neq \{0\}$,

(ii) function $x \mapsto \frac{1}{x}\chi_{(t_0,\infty)}(x)$ belongs to X for some $t_0 > 0$,

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(iii) function $x \mapsto \frac{1}{x}\chi_{(t,\infty)}(x)$ belongs to X for all $t > 0$.

Contents

Theorem [KT17, Th. 16]

Let X be a symmetric space. Then one of the following occurs

(i) $(CX)_a = (CX)_b$ if $X_a \neq \{0\}$,

(ii) $(CX)_a = (CX)_b \cap \Delta_0$ if $X_a = \{0\}$,

where $\Delta_0 = \Delta_0(X) := \{f \in X : \lim_{t \to 0^+} Cf(t) = 0\}$. In particular, if $C : X \to X$ and $X_a \neq \{0\}$, then

$$(CX)_a = C(X_a). \tag{27}$$

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If $C : X \to X$, then

$$(CX)_a = C(X_a) + (CX)_b \cap \Delta_0. \tag{28}$$

Remark [KT17, Remark 19]

$$(Ces_\infty)_a = \begin{cases} \Delta_0 = \{f \in Ces_\infty : \lim_{t\to 0^+} Cf(t) = 0\} & \text{if } I = [0,1] \\ \Delta_0 \cap \Delta_\infty = \{f \in Ces_\infty : \lim_{t\to 0^+,\infty} Cf(t) = 0\} & \text{if } I = [0,\infty), \end{cases}$$

where $\Delta_\infty = \Delta_\infty(X) := \{f \in X : \lim_{t\to\infty} Cf(t) = 0\}$.

[20] [Za83] A. C. Zaanen, *Riesz spaces II*, 1983

Remark [KT17, Remark 19]

$$(Ces_\infty)_a = \begin{cases} \Delta_0 = \{f \in Ces_\infty : \lim_{t\to 0^+} Cf(t) = 0\} & \text{if } I = [0,1] \\ \Delta_0 \cap \Delta_\infty = \{f \in Ces_\infty : \lim_{t\to 0^+,\infty} Cf(t) = 0\} & \text{if } I = [0,\infty), \end{cases}$$

where $\Delta_\infty = \Delta_\infty(X) := \{f \in X : \lim_{t\to\infty} Cf(t) = 0\}$.

- The above conclusion generalizes Zaanen's classical result

$$K_a := (Ces_\infty[0,1])_a = \{f \in K : \lim_{h\to 0^+} \frac{1}{h}\int_0^h |f| \mathrm{d}m = 0\} = \Delta_0(K), \tag{29}$$

where K is alredy mentioned Korenblyum-Kreĭn-Levin space (cf. [Za83, pp. 469-471]). [20]

[20] [Za83] A. C. Zaanen, *Riesz spaces II*, 1983

Theorem [KT17, Prop. 20]

$$(Ces_p)_a = \begin{cases} Ces_p & \text{if } 1 \leqslant p < \infty \\ \{f \in Ces_\infty : \lim_{t \to 0^+, \infty} Cf(t) = 0\} & \text{if } p = \infty. \end{cases}$$

In particular, $Ces_p \in (OC) \iff 1 \leqslant p < \infty$.

- Shiue [Sh70], Hassarda and Husseina [HH73].

Remark [KT17, Remark 13 and Remark 17]

Let X be a symmetric space. If $X_a \neq \{0\}$, then $(CX)_a = C(X_a) \iff \frac{1}{t}\chi_{(\lambda_0,\infty)}(t) \in X_a$ for some $\lambda_0 > 0$ (*for all* $\lambda > 0$).

Remark [KT17, Remark 13 and Remark 17]

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Proposition [KT17, Remark 6]

Let $X[0,1]$ be a symmetric space. $X \neq L^\infty \Leftrightarrow (CX)_a = C(X_a)$.

- Curbera and Ricker [CR16, Prop. 3.1 (c)] (using methods of reprezentation theory and vector measures).

For any Banach function space X we have

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Theorem [KT17, Th. 3], cf. [LM15p, Lemma 1 (a)] & [KK17, Prop. 2]

Let X be a symmetric space such that $C : X \to X$. Then

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Theorem [KT17, Th. 3], cf. [LM15p, Lemma 1 (a)] & [KK17, Prop. 2]

Let X be a symmetric space such that $C : X \to X$. Then

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$X = L^2[0, \frac{1}{4}] \oplus L^\infty[\frac{1}{4}, \frac{1}{2}] \oplus L^2[\frac{1}{2}, 1]$. Then $X \notin (OC)$ but $CX = Ces_2[0, 1]$, see [LM15p]. Therefore $CX \in (OC)$. [a]

[a] [LM15p] K. Leśnik, M. Maligranda, *Interpolation of abstract Cesàro, Copson and Tandori spaces*, 2016

Thank you for your attention