Linear Dynamical Properties of Weighted Backward Shifts on Spaces of Real Analytic Functions

Can Deha Karıksız
(joint work with late Paweł Domański)

Özyeğin University, Istanbul

deha.kariksiz@ozyegin.edu.tr

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Outline

An operator T on a topological vector space X is called

- *hypercyclic* if there is some $x \in X$ such that the set

$$\{x, Tx, T^2x, \cdots, T^nx, \cdots\},$$

called the *orbit* of x under T, is dense in X.

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- *hypercyclic* if there is some $x \in X$ such that the set
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called the *orbit* of x under T, is dense in X.
- *topologically transitive* if for any pair of nonempty open subsets U, V of X, there exists some $n \in \mathbb{N}$ such that
$$T^n(U) \cap V \neq \emptyset.$$

An operator T on a topological vector space X is called

- *mixing* if for any pair of nonempty open subsets U, V of X, there exists some $N \in \mathbb{N}$ such that for every $n \geq N$ we have

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- *mixing* if for any pair of nonempty open subsets U, V of X, there exists some $N \in \mathbb{N}$ such that for every $n \geq N$ we have
$$T^n(U) \cap V \neq \emptyset.$$
- *chaotic* if T is topologically transitive and has a dense set of periodic points.

Let X be a Fréchet sequence space with canonical unit sequences e_n. For a sequence of nonzero scalars $\omega = (\omega_n)_{n\in\mathbb{N}}$, the operator $B_\omega : X \to X$ defined by

$$B_\omega e_n = \omega_n e_{n-1},\ n \geq 1, \quad e_0 = 0,$$

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Theorem

Let $B_\omega : X \to X$ be a weighted backward shift acting on a Fréchet sequence space X in which $(e_n)_{n\in\mathbb{N}}$ is a basis.

(i) *B_ω is hypercyclic $\Leftrightarrow$ there is an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $\left(\prod_{\nu=1}^{n_k} \omega_\nu\right)^{-1} e_{n_k} \to 0$ in X as $k \to \infty$.*

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Theorem

Let $B_\omega : X \to X$ be a weighted backward shift acting on a Fréchet sequence space X in which $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis.

(i) *B_ω is hypercyclic $\Leftrightarrow$ there is an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $\left(\prod_{\nu=1}^{n_k} \omega_\nu\right)^{-1} e_{n_k} \to 0$ in X as $k \to \infty$.*

(ii) *B_ω is mixing $\Leftrightarrow$ $\left(\prod_{\nu=1}^{n} \omega_\nu\right)^{-1} e_n \to 0$ in X as $n \to \infty$.*

(iii) *B_ω is chaotic $\Leftrightarrow$ $\sum_{n=1}^{\infty} \left(\prod_{\nu=1}^{n} \omega_\nu\right)^{-1} e_n$ converges in X.*

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Equivalent Topologies on $A(\Omega)$ (Martineau 1966)

- *Projective limit topology:*

$$A(\Omega) = \operatorname{proj}_{N\in\mathbb{N}} H(K_N) = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}} H^{\infty}(U_{N,n}),$$

where $(K_N)_{N\in\mathbb{N}}$ is a compact increasing exhaustion of Ω, and $(U_{N,n})_{n\in\mathbb{N}}$ are fundamental sequences of neighborhoods of K_N for each N.

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  where $(K_N)_{N\in\mathbb{N}}$ is a compact increasing exhaustion of Ω, and $(U_{N,n})_{n\in\mathbb{N}}$ are fundamental sequences of neighborhoods of K_N for each N.
- *Inductive limit topology:* $A(\Omega) = \mathrm{ind}\, H(U)$ where the inductive limit is taken over all complex neighborhoods of Ω.

Main difficulties

- These locally convex topologies on $A(\Omega)$ are not metrizable, hence $A(\Omega)$ is not a Fréchet space.
- **(Domański, Vogt 2000)** $A(\Omega)$ has no Schauder basis.

Definition

Given a sequence of nonzero scalars $\omega = (\omega_n)_{n\in\mathbb{N}}$, a continuous linear operator

$$B_\omega : A(\Omega) \to A(\Omega),$$

that sends

- the monomials x^n to $\omega_{n-1}x^{n-1}$ for all $n \geq 1$,
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Problem: How to characterize well-defined weighted backward shifts on $A(\Omega)$?

A linear continuous operator $M : A(\Omega) \to A(\Omega)$ is called a *(Hadamard) multiplier* whenever every monomial is an eigenvector.

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Observation

There is a one-to-one correspondence between the weighted backward shifts and the multipliers on $A(\Omega)$ whenever $0 \in \Omega$.

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In this case, we can use the representation theorems for Hadamard multipliers on $A(\Omega)$ **(Domański, Langenbruch 2012)**.

Proposition

Let $\Omega \subset \mathbb{R}$ with $0 \in \Omega$ be an open set. Then, TFAE:

(i) *B_ω is a w.b.s. with the weight sequence $\omega = (\omega_n)$.*

(ii) *B_ω maps a function $\sum_{n=0}^{\infty} f_n z^n$ around zero into a real analytic function on Ω represented around zero by the series $\sum_{n=0}^{\infty} f_n \omega_{n-1} z^{n-1}$.*

Conditions Using Dynamical Transference Principles

An operator T on X is called *quasiconjugate* to an operator S on Y via a continuous map $\phi : Y \to X$ with dense range if

$$T \circ \phi = \phi \circ S.$$

Linear dynamical properties like hypercyclicity, mixing, and chaos are *preserved under quasiconjugacy*.

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By considering B_ω as an operator acting on the space $H(\mathbb{C})$ of entire functions, and the space $H(\{0\})$ of germs of holomorphic functions at zero, we obtain the following two quasiconjugacies.

$$\begin{array}{ccc} H(\mathbb{C}) & \xrightarrow{B_\omega} & H(\mathbb{C}) \\ \downarrow & & \downarrow \\ A(\Omega) & \xrightarrow{B_\omega} & A(\Omega) \end{array} \qquad \begin{array}{ccc} A(\Omega) & \xrightarrow{B_\omega} & A(\Omega) \\ \downarrow & & \downarrow \\ H(\{0\}) & \xrightarrow{B_\omega} & H(\{0\}) \end{array}$$

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Using the quasiconjugacy involving $H(\mathbb{C})$, we obtain the following sufficient conditions.

Proposition (Domański, K. 2018)

Let $\Omega \subset \mathbb{R}$ be an open set with $0 \in \Omega$, and $B_\omega : A(\Omega) \to A(\Omega)$ be a weighted backward shift such that ω has no zero terms.

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(a) *if there is a sequence (n_k) such that for every $R > 0$,*
$$\Big(\prod_{\nu=1}^{n_k} \omega_{\nu-1}\Big)^{-1} R^{n_k} \to 0$$
then B_ω is hypercyclic.

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(b) *if for every $R > 0$, $\left(\prod_{\nu=1}^{n} \omega_{\nu-1}\right)^{-1} R^{n} \to 0$ then B_ω is mixing and chaotic.*

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Using the quasiconjugacy involving $H(\{0\})$ and a construction due to Bonet, we obtain the following necessary conditions.

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(a) *if B_ω is topologically transitive then there is a sequence (n_k) and there is $R > 0$ such that* $$\left(\prod_{\nu=1}^{n_k} \omega_{\nu-1}\right)^{-1} R^{n_k} \to 0.$$

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(b) *if B_ω is mixing then there is $R > 0$ such that* $\left(\prod_{\nu=1}^{n} \omega_{\nu-1}\right)^{-1} R^{n} \to 0.$

Observation

There are weight sequences satisfying the conditions

$$\Big(\prod_{\nu=1}^{n_k}\omega_{\nu-1}\Big)^{-1}R^{n_k}\to 0 \quad \text{or} \quad \Big(\prod_{\nu=1}^{n}\omega_{\nu-1}\Big)^{-1}R^{n}\to 0$$

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Example If we take the *unweighted* backward shift on $A(\Omega)$, that is, $\omega=(\omega_n)$ where $\omega_n=1$ for all $n\in\mathbb{N}$, then

$$\Big(\prod_{\nu=1}^{n}\omega_{\nu-1}\Big)^{-1}R^{n}=R^n\to 0 \text{ only for } 0<R<1.$$

Godefroy-Shapiro criterion

Let T be an operator on a topological vector space X. If the subspaces

$$X_0 := \mathrm{span}\{x \in X : Tx = \lambda x \text{ for some } \lambda \text{ with } |\lambda| < 1\},$$
$$Y_0 := \mathrm{span}\{x \in X : Tx = \lambda x \text{ for some } \lambda \text{ with } |\lambda| > 1\}$$

are dense in X, then T is mixing.
If, moreover, X is a complex space and the subspace

$$Z_0 := \mathrm{span}\{x \in X : Tx = e^{\alpha\pi i} \text{ for some rational } \alpha\}$$

is dense in X, then T is chaotic.

Definition

A set U is called the *star of holomorphy* of a germ $f \in H(\{0\})$ if it is a maximal star-like set in $\mathbb{C}$ around zero such that f extends holomorphically to V.

A set U is *star-like* if for any $z \in U$ we have $\{tz : t \in [0,1]\} \subset U$.

Proposition

Let $\Omega \subseteq \mathbb{R}$ *be an interval with* $0 \in \Omega$, *and* $B_\omega : A(\Omega) \to A(\Omega)$ *be a weighted backward shift such that* ω *has no zero terms.*

If $\sup_n \left(\prod_{\nu=1}^{n} \omega_{\nu-1} \right)^{-1} R^n < \infty$ *for some* $R > 0$, *then*

(i) *zero is an eigenvalue with eigenspace of dimension one,*

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(i) *zero is an eigenvalue with eigenspace of dimension one,*

(ii) *$\lambda \neq 0$ is an eigenvalue of B_ω if and only if $\lambda\Omega$ is contained in the star of holomorphy of the universal eigenfunction*

$$E_\omega(z) := z^{n+1} + \sum_{j=n+2}^{\infty} \left(\prod_{k=n+1}^{j-1} \omega_k \right)^{-1} z^j.$$

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If $\lambda \neq 0$ is an eigenvalue of B_ω, then its eigenspace is one-dimensional, spanned by the function $f_\lambda(z) := E_\omega(\lambda z)$.

Theorem (Domański, K. 2018)

Let $\Omega \subseteq \mathbb{R}$ be an interval with $0 \in \Omega$, and $B_\omega : A(\Omega) \to A(\Omega)$ be a weighted backward shift such that ω has no zero terms.

(a) *If B_ω has a universal eigenfunction such that its star of holomorphy contains at least one interval $\lambda\Omega$ for some $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, then B_ω is mixing.*

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(b) *If additionally to (a) the star of holomorphy of the universal eigenfunction contains a strip $\{z \in \mathbb{C} : |z - t\lambda| < \varepsilon \text{ for some } t \in \mathbb{R}\}$ for some $\varepsilon > 0$, then B_ω is also hypercyclic.*

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(c) *If additionally to (a) the star of holomorphy of the universal eigenfunction contains a set of the form $\{te^{\pi ai} : t \in \Omega, a \in (a_1, a_2)\}$ for some $-\infty < a_1 < a_2 < \infty$ then B_ω is also chaotic.*

Examples

Let Ω be an interval containing zero. The following weighted backward shifts are chaotic, mixing, and hypercyclic.

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(i) the unweighted shift $B_{\mathbf{1}}$,

(ii) the differentiation operator $D : A(\Omega) \to A(\Omega)$ which corresponds to the weighted shift with $\omega_n = n + 1$,

(iii) the weighted backward shift $Q_k : A(\Omega) \to A(\Omega)$, $k > 0$, $Q_k(f)(z) = \int_0^1 f'(zt)t^k \, dt$ which corresponds to the weighted backward shift with $\omega_n = \frac{n+1}{n+1+k}$.

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