Dynamics of weighted composition operators on function spaces defined by local properties

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Continuous linear operator T on a lcs E is called

- *(topologically) transitive* :$\Leftrightarrow$
  $\forall\, U, V \subseteq E$ open, non-empty $\exists\, m \in \mathbb{N}_0 : T^m(U) \cap V \neq \emptyset$
  (E separable Fréchet, equivalent to T *hypercyclic*, i.e. there is $x \in E$ s.th. $\{T^m x;\, m \in \mathbb{N}_0\}$ is dense in E.)

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- *weakly mixing* $:\Leftrightarrow T \oplus T$ transitive on $E \oplus E$, i.e.
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- *power bounded* :⇔ $\{T^m;\, m \in \mathbb{N}_0\}$ is equicontinuous, i.e. $\forall\, p \in cs(E)\, \exists\, q \in cs(E)\, \forall\, m \in \mathbb{N}_0, x \in E : p(T^m x) \leq q(x)$

Albanese, Bonet, Ricker '09: E Fréchet-Montel, T power bounded $\Rightarrow T$ *uniformly mean ergodic*, i.e. $\forall\, x \in E\, \exists\, \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} T^m x$ and convergence is uniform on bounded subsets of E.

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Folklore (see e.g. Yoshida, 1980): E sequentially complete lcs,
T continuous, linear, power bounded $\Rightarrow$ $(\exp(sT))_{s\geq 0}$ C_0-semigroup,
where $\exp(sT)x = \sum_{m=0}^{\infty} \frac{s^m}{m!} T^m x$

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Objective: study these dynamical properties for weighted composition operators $C_{w,\psi}(f) = w \cdot (f \circ \psi)$ on function spaces "in a general framework".

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$\forall\, X \subseteq \Omega$ open: $\mathscr{F}(X)$ is a $\mathbb{K}$-vector space of $\mathbb{K}$-valued functions on X s.th.

- $\forall\, Y \subseteq X \subseteq \Omega$ open: $r_X^Y : \mathscr{F}(X) \to \mathscr{F}(Y), f \mapsto f_{|Y}$ well-defined
- (Gluing) $\forall$ open cover $(X_\iota)_{\iota\in I}$ of an open set $X \subseteq \Omega$
  $\forall\, (f_\iota)_{\iota\in I} \in \prod_{\iota\in I} \mathscr{F}(X_\iota)$ with $f_{\iota|X_\iota\cap X_\kappa} = f_{\kappa|X_\iota\cap X_\kappa}\, (\iota,\kappa \in I)$ there is $f \in \mathscr{F}(X)$ with $f_{|X_\iota} = f_\iota\, (\iota \in I)$.

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$\Rightarrow\ \forall\, X \subseteq \Omega$ open $\forall\,(X_n)_{n\in\mathbb{N}_0}$ open, relatively compact exhaustion of X:

$$\begin{aligned} \mathscr{F}(X) &\cong \text{proj}(\mathscr{F}(X_{n+1}), r_{X_{n+1}}^{X_n})_{n\in\mathbb{N}_0} \\ &= \{(f_n)_{n\in\mathbb{N}_0} \in \prod_n \mathscr{F}(X_n);\ \forall\, n \in \mathbb{N}:\ f_{n|X_{n-1}} = f_{n-1}\} \\ \text{via } f &\mapsto (f_{|X_n})_{n\in\mathbb{N}_0} \end{aligned}$$

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Examples: $\Omega = \mathbb{R}^d$, $\mathscr{F}(X) = C^\infty(X)$, $\mathscr{F}(X) = C(X)$, $\mathscr{F}(X) = \mathscr{A}(X)$, or for $\Omega = \mathbb{C}$, $\mathscr{F}(X) = \mathscr{H}(X)$. $L^p(X)$ is not a sheaf.

We define the following properties for a sheaf of functions $\mathscr{F}$ on Ω:

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Examples:

- $\mathscr{F} = C^r$ $(r \in \mathbb{N}_0 \cup \{\infty\})$ on $\Omega = \mathbb{R}^d$ satisfies $(\mathscr{F}1) - (\mathscr{F}3)$ when equipped with the seminorms $\|f\|_{l,K} := \max_{|\alpha|\leq l, x\in K} |\partial^\alpha f(x)|$ $(l < r+1, K \Subset X)$.

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Additionally, in case of $(\mathscr{F}1)$, $\Omega = \mathbb{R}^d$ and $\mathscr{F}(X) \subseteq C^1(X)$ we define

$(\mathscr{F}4)$ $\forall\, X \subseteq \mathbb{R}^d$ open, $1 \leq j \leq d$, $x \in X :\ f \mapsto \partial_j f(x) \in \mathscr{F}(X)'$ and for each $h \in \mathbb{R}^d \backslash \{0\}, \lambda \in \mathbb{K}$:

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An almost characterisation of weak mixing

Assume $\mathscr{F}$ satisfies $(\mathscr{F}1)-(\mathscr{F}3)$. If $C_{w,\psi}$ acts locally on $\mathscr{F}(X)$, $i) \Rightarrow ii) \Rightarrow iv)$:

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b) $\exists\, (X_n)_{n\in\mathbb{N}_0}\, \forall\, n \in \mathbb{N}_0\, \exists\, m \in \mathbb{N}$:

b1) $X_n \cap \psi^m(X_n) = \emptyset$, b2) $\psi^m(X_n)$ is open, b3) $r_X^{X_n \cup \psi^m(X_n)}$ has dense range.

An almost characterisation of weak mixing

Assume $\mathscr{F}$ satisfies $(\mathscr{F}1)-(\mathscr{F}3)$. If $C_{w,\psi}$ acts locally on $\mathscr{F}(X)$, $i) \Rightarrow ii) \Rightarrow iv)$:

i) a) $\forall\, m \in \mathbb{N}_0, Y \subseteq \psi^m(X)$ open, rel. comp. $r_X^{(\psi^m)^{-1}(Y)}(\mathscr{F}(X)) \subseteq \overline{\left(C_{w,\psi,(\psi^{(m-1)})^{-1}(Y)} \circ \ldots \circ C_{w,\psi,Y}\right)(\mathscr{F}(Y))}^{\mathscr{F}\left((\psi^m)^{-1}(Y)\right)}$

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ii) $C_{w,\psi}$ is weakly mixing on $\mathscr{F}(X)$.

iv) a) from i) holds, w has no zeros, ψ is injective and run-away [and in case of $(\mathscr{F}4)$ and ψ C^1, additionally $\det J\psi(x) \neq 0, x \in X$].

An almost characterisation of weak mixing/transitivity

Assume $\mathscr{F}$ satisfies $(\mathscr{F}1)-(\mathscr{F}3)$. If $C_{w,\psi}$ acts locally on $\mathscr{F}(X)$, $i) \Rightarrow ii) \Rightarrow iv)$:

i) a) $\forall\, m \in \mathbb{N}_0, Y \subseteq \psi^m(X)$ open, rel. comp. $r_X^{(\psi^m)^{-1}(Y)}(\mathscr{F}(X)) \subseteq \overline{\Big(C_{w,\psi,(\psi^{(m-1)})^{-1}(Y)} \circ \ldots \circ C_{w,\psi,Y}\Big)(\mathscr{F}(Y))}^{\mathscr{F}\left((\psi^m)^{-1}(Y)\right)}$

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ii) $C_{w,\psi}$ is weakly mixing on $\mathscr{F}(X)$.

iii) $C_{w,\psi}$ is transitive on $\mathscr{F}(X)$.

iv) a) from i) holds, w has no zeros, ψ is injective and run-away [and in case of $(\mathscr{F}4)$ and ψ C^1, additionally det$J\psi(x) \neq 0, x \in X$].

Additionally, if $\mathscr{F}(\Omega)$ is dense in $C(\Omega)$ or if $|w(x)| \leq 1, x \in X$, then it holds $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv)$.

Some thoughts on the condition a) $\forall\, m \in \mathbb{N}_0, Y \subseteq \psi^m(X)$ open, rel. comp.

$$r_X^{(\psi^m)^{-1}(Y)}(\mathscr{F}(X)) \subseteq \overline{\left(C_{w,\psi,(\psi^{(m-1)})^{-1}(Y)} \circ \ldots \circ C_{w,\psi,Y}\right)(\mathscr{F}(Y))}^{\mathscr{F}((\psi^m)^{-1}(Y))}$$

for injective, open ψ (with $\det J\psi(x) \neq 0$) and w without zeros:

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For $f \in \mathscr{F}((\psi^m)^{-1}(Y))$,

$$\tilde{f} : Y \to \mathbb{K}, y \mapsto \Big(\frac{f}{\prod_{j=0}^{m-1} w(\psi^j(\cdot))}\Big) \circ (\psi^m)^{-1}(y)$$

is well-defined and continuous. In case of $\tilde{f} \in \mathscr{F}(Y)$, we have

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$\tilde{f} \in \mathscr{F}(Y)$ in many concrete cases for $\mathscr{F}$ whenever w has no zeros and ψ is injective (and open); for example: $\mathscr{F} =$ continuous ($X \subseteq \mathbb{R}^d$ open), smooth, holomorphic, or real analytic functions

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Moreover: $C_{w,\psi}$ has dense range $\Rightarrow$ condition satisfied (without any restrictions on w and ψ)

$\psi : X \to X$ has *stable orbits* $:\Leftrightarrow \forall\, K \Subset X\, \exists\, L \Subset X\, \forall\, m \in \mathbb{N}_0 :\ \psi^m(K) \subseteq L$

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Power boundedness

Assume that $\mathscr{F}$ satisfies $(\mathscr{F}1)$. Moreover, assume

a) $\forall\, x \in X :$ kern $\delta_x \neq \mathscr{F}(X)$.

b) $\exists\,(X_n)_{n\in\mathbb{N}_0}\,\forall\, n, x \in X\backslash\overline{X}_n, W \subseteq X\backslash\overline{X}_n$ nbh. of $x\ \exists\, U \subseteq W$ open nbh. x: $r_X^{X_n\cup U}$ has dense range.

c) $\forall\, m \in \mathbb{N}_0 :\ \{x \in X;\, w(\psi^m(x)) \neq 0\}$ is dense in X.

If $w \in \mathscr{F}(X)$, $i) \Rightarrow ii)$:

i) $C_{w,\psi}$ is power bounded on $\mathscr{F}(X)$.

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c) satisfied if $\overline{w^{-1}(\mathbb{K}\backslash\{0\})} = X$ and $\forall\, x\, \exists$ open nbh. $U_x :\ \psi_{|U_x}$ injective, open.

Apply (almost) characterisations of transitivity/ (weak) mixing and power boundedness to kernels of differential operators:

Let $P \in \mathbb{C}[X_1, \ldots, X_d], P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ be non-constant, $X \subseteq \mathbb{R}^d$ open
$\mathscr{E}_P(X) := \{f \in C^\infty(X);\ P(\partial)f = 0 \text{ in } X\}$, where $P(\partial)f = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha f$.

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$\mathscr{E}_P(X)$ is a closed subspace of $C^\infty(X)$, hence a separable nuclear Fréchet space (thus Montel), and $\mathscr{F} = \mathscr{E}_P$ defines sheaf on $\mathbb{R}^d$ satisfying $(\mathscr{F}1)$ and $(\mathscr{F}2)$.

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$(\mathscr{F}3)$ need not be satisfied: $d = 2, P(\xi_1, \xi_2) = \xi_1 \Rightarrow \mathscr{E}_P$ does not separate points in $\mathbb{R}^2$.

Concrete examples:

- $\mathbb{R}^2 = \mathbb{C}$ and $P(\xi_1, \xi_2) = \frac{1}{2}(\xi_1 + i\xi_2) \Rightarrow P(\partial) = \overline{\partial}$ Cauchy-Riemann operator and $\mathscr{E}_P(X) = \mathscr{H}(X)$
- $P(\xi_1, \dots, \xi_d) = \sum_{j=1}^d \xi_j^2 \Rightarrow P(\partial) = \triangle$ Laplace operator, $\mathscr{E}_P(X) = \{f; f \text{ harmonic in } X\}$
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P *hypoelliptic* $:\Leftrightarrow \forall\, u \in \mathscr{D}'(X) : \big(P(\partial)u = 0 \Rightarrow u \in C^\infty(X)\big)$
Then $\mathscr{E}_P(X) = \{u \in \mathscr{D}'(X);\, P(\partial)u = 0\}$, the relative top. from $\mathscr{D}'(X)$ on $\mathscr{E}_P(X)$ coincides with the original top. on $\mathscr{E}_P(X)$, and therefore: $\mathscr{E}_P(X)$ is a nuclear Fréchet space when equipped with the compact open topology.

P elliptic $\Rightarrow$ P hypoelliptic $\Rightarrow$ $\mathscr{E}_P$ satisfies $(\mathscr{F}1) - (\mathscr{F}4)$

Objective: Apply (almost) characterisation of transitivity/weak mixing and power boundedness of $C_{w,\psi}$ to $\mathscr{E}_P$.

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To apply both (almost) characterisations we need particular open, rel. comp. exh. $(X_n)_{n\in\mathbb{N}_0}$ of open $X \subseteq \mathbb{R}^d$ open such that $r_X^{X_n \cup B}$ has dense range for all n and suitable $B \subseteq X \backslash \overline{X}_n$ open.

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This implies that necessarily $P(\partial):C^\infty(X)\to C^\infty(X)$ is surjective ($\Leftrightarrow$: X is P-convex).

True for elliptic operators and arbitrary open $X\subseteq\mathbb{R}^d$ but not true for general (hypoelliptic) operators and arbitrary X.

Theorem (Power boundedness on $\mathscr{E}_P$)

Let $X \subseteq \mathbb{R}^d$ be open, $w \in C^\infty(X), \psi : X \to X$ smooth s.th. $w^{-1}(\mathbb{C}\backslash\{0\})$ is dense in X and ψ is locally injective, P hypoelliptic with $P(0) = 0$. Then in any of the two cases

a) P is elliptic,

b) $d \geq 3$, $\{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\}$ is a one-dimensional subspace, X is P-convex,

tfae:

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If $w = 1$ the following are equivalent to i), ii).

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iii) $C_{w,\psi}$ is (uniformly) mean ergodic on $\mathscr{E}_P(X)$.

b) applicable to non-degenerate parabolic operators like the heat operator

Theorem (Weak mixing on $\mathscr{E}_P$)

Let $X \subseteq \mathbb{R}^d$ be homeomorphic to $\mathbb{R}^d$ and let P be elliptic. If $C_{w,\psi}$ acts locally on $\mathscr{E}_P(X)$, tfae:

i) $C_{w,\psi}$ is weakly mixing on $\mathscr{E}_P(X)$.

ii) $C_{w,\psi}$ has dense range, w has no zeros, and ψ is injective and run-away.

Moreover, $\det J\psi(x) \neq 0$ for all $x \in X$ can be added to ii).

Theorem (Weak mixing on $\mathscr{E}_P$)

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Moreover, det$J\psi(x) \neq 0$ for all $x \in X$ can be added to ii). Additionally, if $|w| \leq 1$ the above are equivalent to

iii) $C_{w,\psi}$ is hypercyclic on $\mathscr{E}_P(X)$.

Consider the elliptic operator $P(\partial) = \triangle - \lambda, \lambda \in \mathbb{C}$.

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Given $X \subseteq \mathbb{R}^d$ open and C^2-functions w, ψ. $C_{w,\psi}$ is well-defined on $\mathscr{E}_P(X)$ if and only if

- $\forall\, 1 \leq j \neq k \leq d:\ w|\nabla\psi_j|^2 = w|\nabla\psi_k|^2, w\langle\nabla\psi_j, \nabla\psi_k\rangle = 0$
- $\forall\, 1 \leq j \leq d:\ w\,\Delta\psi_j + 2\langle\nabla w, \nabla\psi_j\rangle = 0$
- $\Delta w - \lambda w = -\lambda w|\nabla\psi_1|^2$

($\Rightarrow$ $C_{w,\psi}$ acts locally on $\mathscr{E}_P(X)$)

Corollary

Let $X \subseteq \mathbb{R}^d$ be open and homeomorphic to $\mathbb{R}^d$, $P(\partial) = \triangle - \lambda, \lambda \in \mathbb{C}$, w, ψ be C^2 such that $C_{w,\psi}$ is well-defined on $\mathscr{E}_P(X)$. Tfae

- i) $C_{w,\psi}$ is hypercyclic on $\mathscr{E}_P(X)$.
- ii) $C_{w,\psi}$ is weakly mixing on $\mathscr{E}_P(X)$.
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Question: When is an arbitrary (injective) ψ satisfying

- $\forall\, 1 \leq j \neq k \leq d:\; w|\nabla\psi_j|^2 = w|\nabla\psi_k|^2, w\langle\nabla\psi_j, \nabla\psi_k\rangle = 0$
- $\forall\, 1 \leq j \leq d:\; w\,\triangle\psi_j + 2\langle\nabla w, \nabla\psi_j\rangle = 0$
- $\triangle w - \lambda w = \lambda w|\nabla\psi_1|^2$

for a zero-free w run-away, resp. when does it have stable orbits?

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