Global pseudo-differential operators in ultradifferentiable classes

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- Compare this definition with other local or classical versions of pseudo-differential operators.
- Study the transpose, the composition of such operators. Give sufficient conditions for the existence of parametrices and apply it to the study of elliptic problems.

Basic ideas

Roughly speaking, a pseudo-differential operator is a mapping $f \mapsto Tf$ given by

$$(Tf)(x) = \int_{\mathbb{R}^d} a(x,\xi)\widehat{f}(\xi)e^{i x\cdot\xi} d\xi,$$

where $\widehat{f}$ is the Fourier transform of f and $a(x,\xi)$ is the *symbol* of T.

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In other words, using the definition of Fourier transform, we obtain the iterated integral

$$(Tf)(x) = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,\xi) f(y)dy \Big) d\xi,$$

and the function $a(x,\xi)$ could depend also on a third variable y, $a(x,y,\xi)$ and is called *amplitude* of T.

- A linear partial differential operator with variable coefficients $P(x, D) = \sum_{|\alpha|\leqslant m} a_\alpha(x) D^\alpha$, $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$, can be written as above with symbol

$$a(x,\xi) = \frac{1}{(2\pi)^p} p(x,\xi),$$

where $p(x,\xi) = \sum_{|\alpha|\leqslant m} a_\alpha(x)\xi^\alpha$ is the characteristic polynomial of $P(x, D)$.

Basic examples in the local theory

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- For every elliptic linear partial differential operator with constant coefficients $P(D)$, the symbol

$$a(\xi)=\frac{1}{(2\pi)^p p(\xi)},$$

  for $|\xi|$ big enough, gives a parametrix T for $P(D)$.

Classical symbol classes

Classical, local case

Hörmander, 1965: for $0 \leqslant \delta < \rho \leqslant 1$ we take $p(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^d)$ such that for every $K \subset\subset \Omega$, $\alpha, \beta \in \mathbb{N}_0^d$ there is $C = C_{K,\alpha,\beta} > 0$ with

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leqslant C(1 + |\xi|)^{m-\rho|\beta|+\delta|\alpha|}, \quad x \in K, \xi \in \mathbb{R}^d.$$

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Classical, global case

Shubin, 1971: for $0 < \rho \leqslant 1$ we consider $a(x,\xi) \in C^\infty(\mathbb{R}^{2d})$ such that for every $\alpha, \beta \in \mathbb{N}_0^d$ there is $C := C_{\alpha,\beta} > 0$ with

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant C\langle (x,\xi)\rangle^{m-\rho|\alpha+\beta|}, \quad (x,\xi) \in \mathbb{R}^{2d},$$

where $\langle z \rangle = (1+|z|^2)^{1/2}$, $z \in \mathbb{R}^d$.

Symbols of infinite order

Among other authors (like Boutet de Monvel, Fernández-Galbis-Jornet, Cappiello, Rodino,...) we mention

Local Gevrey setting

Zanghirati, 1985: for $0 \leqslant \delta < \rho \leqslant 1$ we take $p(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^d)$ such that for every $K \subset\subset \Omega$ there are $C, B > 0$ such that for every $\varepsilon > 0$ there is $C_\varepsilon > 0$ with

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leqslant C_\varepsilon C^{|\alpha+\beta|} (\alpha!)^{s(\rho-\delta)} \beta! (1+|\xi|)^{\delta|\alpha|-\rho|\beta|} e^{\varepsilon|\xi|^{1/s}},$$

for each $\alpha, \beta \in \mathbb{N}_0^d$, $x \in K$, $1 + |\xi| \geqslant B|\beta|^s$.

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for each $\alpha, \beta \in \mathbb{N}_0^d$, $x \in K$, $1+|\xi| \geqslant B|\beta|^s$.

Global setting

Nicola-Rodino (a general kind of finite order symbols), Cappiello, Rodino (Gelfand-Shilov classes), Cappiello, Pilipovic, Prangoski (classes of tempered ultradifferentiable functions in the sense of Komatsu).

Our setting

Weight functions in the sense of Braun-Meise-Taylor

A *non-quasianalytic weight function* is a continuous increasing function $\omega : [0,+\infty) \to [0,+\infty)$ satisfying the following properties:

(α) $\exists\, L > 0$ s.t. $\omega(2t) \leq L(\omega(t)+1)\ \forall t \geq 0$;

(β) $\displaystyle\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty$;

(γ) $\exists\, a \in \mathbb{R},\ b > 0$ s.t. $\quad \omega(t) \geq a + b\log(1+t) \qquad \forall t \geq 0$;

(δ) $\varphi_\omega :\ t \mapsto \omega(e^t)$ is convex.

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We then define $\omega(\xi) = \omega(|\xi|)$ for $\xi \in \mathbb{C}^d$.

The Young conjugate φ^* of φ is defined by

$$\varphi^*(s) := \sup_{t\ge 0}\{st - \varphi(t)\},$$

for all $s \ge 0$.

Definition (Björck's class)

For a weight ω as before we define $\mathcal{S}_\omega(\mathbb{R}^n)$ as the set of all $u \in C^\infty(\mathbb{R}^d)$ such that for each $\lambda > 0$, there is $C_\lambda > 0$ such that

$$\sup_{\alpha,\beta\in\mathbb{N}_0^n} \sup_{x\in\mathbb{R}^n} |x^\beta D^\alpha u(x)| e^{-\lambda\varphi^*\left(\frac{|\alpha+\beta|}{\lambda}\right)} \leq C_\lambda.$$

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Useful characterization

The last condition can be replaced by: for each $\lambda, \mu > 0$ there is $C_{\lambda,\mu} > 0$ such that

$$\sup_{\alpha\in\mathbb{N}_0^n} \sup_{x\in\mathbb{R}^n} |D^\alpha u(x)| e^{-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)} e^{\mu\omega(x)} \leq C_{\lambda,\mu}.$$

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As usual, the corresponding dual space is denoted by $\mathcal{S}'_\omega(\mathbb{R}^n)$.

Pseudo-differential operators

Definition (Following Prangoski (2013))

An amplitude in $\mathrm{GA}_\rho^{m,\omega}(\mathbb{R}^{3d})$ (of order m) is a function $a(x,y,\xi) \in C^\infty(\mathbb{R}^{3d})$ such that for all $n \in \mathbb{N}$ there is $C_n > 0$ with

$$\left|D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi)\right| \leq C_n \frac{\langle x-y\rangle^{\rho|\alpha+\beta+\gamma|}}{\langle (x,y,\xi)\rangle^{\rho|\alpha+\beta+\gamma|}} \frac{e^{\rho n \varphi^*\left(\frac{|\alpha+\beta+\gamma|}{n}\right)}}{e^{-m(\omega(|x|)+\omega(|y|)+\omega(|\xi|))}}$$

for all $(x,y,\xi) \in \mathbb{R}^{3d}$ and $(\alpha,\beta,\gamma) \in \mathbb{N}_0^{3d}$.

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We want to see that the operator given by the iterated integral

$$Af(x) = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,y,\xi) f(y)dy\right)d\xi, \quad f \in \mathcal{S}_\omega(\mathbb{R}^d),$$

acting $A : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ is well defined and continuous.

Idea to see that it is well defined

- We represent the operator as an oscillatory integral: Let $\chi \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ with $\chi(0,0) = 1$, and consider, $f \in \mathcal{S}_\omega, \ \delta > 0$,

$$I_{\delta,\chi}(f)(x) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} a(x,y,\xi)\chi(\delta x, \delta\xi) f(y) dy d\xi.$$

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- To see that $I_{\delta,\chi} : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ is well defined and continuous:

Theorem (Braun-Langenbruch)

There exists an entire function $G(z) = \sum_{\alpha \in \mathbb{N}_0^d} a_\alpha z^\alpha \in \mathcal{H}(\mathbb{C}^d)$ such that for some constants $C_1, C_2, C_3, C_4 > 0$:

i') $\log|G(z)| \leq \omega(|z|) + C_1, \quad \forall z \in \mathbb{C}^d$;

ii') $\log|G(z)| \geq C_2\omega(|z|) - C_4, \quad$ *for* $z \in \widetilde{U} := \{z \in \mathbb{C}^d : |\mathrm{Im}(z)| \leq C_3(|\mathrm{Re}(z)| + 1)\}$.

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- We consider the ultradifferential operator $G(D)$ associated to $G(z)$ and integrate by parts in $I_{\delta,\chi}(f)(x)$ with the formula

$$e^{i(x-y)\xi} = \frac{1}{G(y-x)} G(-D_\xi)\Big(\frac{1}{G(\xi)} G(-D_y) e^{i(x-y)\xi}\Big).$$

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- Since $\mathcal{S}_\omega(\mathbb{R}^d)$ is a Fréchet space, by Lebesgue theorem, the pseudo-differential operator $A : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ is well defined and continuous and, moreover,

$$A(f) = \mathcal{S}_\omega(\mathbb{R}^d) - \lim_{\delta\to 0+} I_{\delta,\chi}(f), \quad f \in \mathcal{S}_\omega(\mathbb{R}^d).$$

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- By duality, the operator A can be extended to a linear and continuous operator $\widetilde{A} : \mathcal{S}'_\omega(\mathbb{R}^d) \to \mathcal{S}'_\omega(\mathbb{R}^d)$

Example

- For the limit case $\omega(t) = \log(1 + t^2)$, we have $\mathcal{S}_\omega(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$ and we recover the class of symbols of finite order of Shubin type.

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- A linear partial differential operator $P(x,D) = \sum_{|\alpha|\leqslant m} a_\alpha(x) D^\alpha$ is a pseudo-differential operator with symbol (amplitude) in $GA_1^{k,\omega}$ when the coefficients a_α are *"regular enough"*.

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- For the Gevrey case $\omega(t) = t^{1/s}$ we recover symbols suitable for some Gelfand-Shilov spaces.

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- A function in $\mathcal{S}_\omega(\mathbb{R}^{2d})$ with compact support is a symbol of order 0 in $\mathbb{R}^{2d}$.
- If $G \in \mathcal{H}(\mathbb{C}^d)$ with $\log|G(z)| = O(\omega(z))$ as $|z| \to \infty$, then $G|_{\mathbb{R}^{2d}}$ is a symbol of some order $m > 0$.

Symbolic calculus

Formal sums

$FGS_\rho^{m,\omega}(\mathbb{R}^{2d})$ the set of all formal sums $\sum_{j\in\mathbb{N}_0} a_j(x,\xi)$ such that $a_j(x,\xi) \in C^\infty(\mathbb{R}^{2d})$ and there is $R \geq 1$ such that for each $n \in \mathbb{N}$ there exists $C_n > 0$ with

$$\left|D_x^\alpha D_\xi^\beta a_j(x,\xi)\right| \leq C_n \left(\frac{1}{\langle (x,\xi)\rangle}\right)^{\rho(|\alpha+\beta|+j)} e^{\rho n\varphi^*\left(\frac{|\alpha+\beta|+j}{n}\right)} e^{m\omega(x)} e^{m\omega(\xi)}$$

for each $j \in \mathbb{N}_0$, $(\alpha,\beta) \in \mathbb{N}_0^{2d}$ and $\log\left(\frac{\langle (x,\xi)\rangle}{R}\right) \geq \frac{n}{j}\varphi^*\left(\frac{j}{n}\right)$.

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for each $j\in\mathbb{N}_0$, $(\alpha,\beta)\in\mathbb{N}_0^{2d}$ and $\log\left(\frac{\langle (x,\xi)\rangle}{R}\right)\geq \frac{n}{j}\varphi^*\left(\frac{j}{n}\right)$.

Then, $\sum a_j\sim\sum b_j$, if there is $R\geq 1$ such that for each $n\in\mathbb{N}$ there exist $C_n>0$ and $N_n\in\mathbb{N}$ with

$$\left|D_x^\alpha D_\xi^\beta \sum_{j<N}(a_j-b_j)\right| \leq C_n \frac{e^{\rho n\varphi^*\left(\frac{|\alpha+\beta|+N}{n}\right)}}{\langle (x,\xi)\rangle^{\rho(|\alpha+\beta|+N)}} e^{m\omega(x)+m\omega(\xi)},$$

for every $N\geq N_n$, $(\alpha,\beta)\in\mathbb{N}_0^{2d}$ and $\log\left(\frac{\langle (x,\xi)\rangle}{R}\right)\geq \frac{n}{N}\varphi^*\left(\frac{N}{n}\right)$.

Composition of two operators

For any $k > 0$, we denote

$$\Delta_k := \left\{(x, y) \in \mathbb{R}^{2d} : |x - y| < k\right\}.$$

Theorem ("Pseudo-local property")

Given $k > 0$ and $a(x, y, \xi) \in \mathrm{GA}_{\rho}^{m,\omega}$, we have that the formal kernel

$$K(x, y) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} a(x, y, \xi) d\xi$$

satisfies

1. $K(x, y) \in C^{\infty}\left(\mathbb{R}^{2d} \setminus \overline{\Delta_k}\right)$,
2. *For every $\lambda > 0$, there exists $C_\lambda > 0$ such that for all $(x, y) \in \mathbb{R}^{2d} \setminus \Delta_k$ and all $\alpha, \beta \in \mathbb{N}_0^d$*

$$|D_x^{\alpha} D_{\xi}^{\beta} K(x, y)| \leq C_\lambda e^{\lambda \varphi^* \left(\frac{|\alpha+\beta|}{\lambda}\right)} e^{-\lambda\omega(x)} e^{-\lambda\omega(y)}.$$

Composition of two operators II

An operator acting

$$R : \mathcal{S}'_\omega(\mathbb{R}^{2d}) \to \mathcal{S}_\omega(\mathbb{R}^{2d})$$

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Theorem

Let $a(x, y, \xi) \in \mathrm{GA}_\rho^{m,\omega}$ with pseudo-differential operator associated A. Then there is a pseudo-differential operator P given by a symbol $p(x, \xi)$and a ω-regularizing operator R such that

$$Au = Pu + Ru, \quad u \in \mathcal{S}_\omega,$$

and, moreover,

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_j(x, \xi) = \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \frac{1}{\alpha!} D_\xi^\alpha \partial_y^\alpha a(x, y, \xi) \mid_{y=x} .$$

Theorem

Let $p(x,\xi), q(x,\xi) \in \mathrm{GS}^{m,\omega}$ and let $P, Q : \mathcal{S}_\omega \to \mathcal{S}_\omega$ be their corresponding pseudo-differential operators. Then, modulus an ω-regularizing operator, we can define its composition $P \circ Q : \mathcal{S}_\omega \to \mathcal{S}_\omega$ in such a way that it coincides with the pseudo-differential operator associated to the formal sum

$$(2\pi)^d \big(p(x,\xi) \circ q(x,\xi)\big) := (2\pi)^d \Big(p(x,\xi)q(x,\xi) + \sum_{j>0} \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi)\Big).$$

Existence of parametrices

Theorem

Let ω be a weight and σ a sub-additive weight with $\omega(t^{1/\rho}) = o\big(\sigma(t)\big)$ as $t \to \infty$. Let $p(x,\xi) \in \mathrm{GS}_\rho^{m,\omega}$ such that for some $R > 0$:

(i) $|p(x,\xi)| \geq \frac{1}{R} e^{-m\omega(x)} e^{-m\omega(\xi)}$ *for* $\langle (x,\xi) \rangle \geqslant R$*, and*

(ii) *There are $n \in \mathbb{N}$ and $C > 0$ such that for* $\langle (x,\xi) \rangle \geqslant R$,

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C^{|\alpha+\beta|} \frac{e^{\frac{1}{n}\varphi_\sigma^*(n|\alpha|)} e^{\frac{1}{n}\varphi_\sigma^*(n|\beta|)}}{\langle (x,\xi) \rangle^{|\alpha+\beta|}} |p(x,\xi)|.$$

Then, there is $q(x,\xi) \in \mathrm{GS}_\rho^{m,\omega}$ such that $q \circ p \sim 1$ in $\mathrm{FGS}_\rho^{m,\omega}$.

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(ii) *There are $n \in \mathbb{N}$ and $C > 0$ such that for $\langle (x,\xi) \rangle \geqslant R$,*

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C^{|\alpha+\beta|} \frac{e^{\frac{1}{n}\varphi_\sigma^*(n|\alpha|)} e^{\frac{1}{n}\varphi_\sigma^*(n|\beta|)}}{\langle (x,\xi) \rangle^{|\alpha+\beta|}} |p(x,\xi)|.$$

Then, there is $q(x,\xi) \in \mathrm{GS}_\rho^{m,\omega}$ such that $q \circ p \sim 1$ in $\mathrm{FGS}_\rho^{m,\omega}$.

Corollary

If $p(x,\xi)$ satisfies (i) and (ii), we have

$$Pu = f \in \mathcal{S}_\omega(\mathbb{R}^d),\ u \in \mathcal{S}'_\omega(\mathbb{R}^d) \Rightarrow u \in \mathcal{S}_\omega(\mathbb{R}^d).$$