Composition operators on the Schwartz space

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Joint work with
Carmen Fernández and Antonio Galbis

The talk is based on

- A. Galbis, E. Jordá; *Composition operators on the Schwartz space*, Rev. Mat. Iberoam. 34 (2018), no. 1, 397–412.
- C. Fernández, A. Galbis, E. Jordá; *Dynamics and spectra of composition operators on the Schwartz space*, J. Funct. Anal. 274 (2018), no. 12, 3503–3530.
- C. Fernández, A. Galbis, E. Jordá; *Spectra of composition operators on the Schwartz space*, work in progress.

Some history

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function. Then $C_\varphi : C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^n),\ f \mapsto f \circ \varphi$.

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- Kenessey and Wengenroth. Closed range property of $C_\varphi : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R})$ for injective and smooth $\varphi : \mathbb{R} \to \mathbb{R}^d$

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- Przestacki. Dynamics of composition operators on $C^\infty(\mathbb{R})$
- Bonet, Domański. Dynamics and spectrum of composition operators on real analytic functions
- Kalmes. Dynamics of weighted composition operators in some spaces of smooth functions.

Aim

$S(\mathbb{R})$ consists of those smooth functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$\pi_n(f) := \sup_{x\in\mathbb{R}} \sup_{1\leqslant j\leqslant n} (1 + x^2)^n |f^{(j)}(x)| < \infty$$

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- Dynamics and spectra of composition operators.

Symbols

Definition

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$\varphi \in C^\infty(\mathbb{R})$ is a symbol for $S(\mathbb{R})$ if and only if the following conditions are satisfied:

(i) *For all $j \in \mathbb{N}_0$ there exist $C, p > 0$ such that*

$$\left|\varphi^{(j)}(x)\right| \leqslant C(1 + \varphi(x)^2)^p \quad \forall x \in \mathbb{R}.$$

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Let φ be a symbol for$S(\mathbb{R})$

- $C_\varphi : S(\mathbb{R}) \to S(\mathbb{R}),\ f \mapsto f \circ \varphi$, is continuous
- $C_\varphi : S(\mathbb{R}) \to S(\mathbb{R}),\ f \mapsto f \circ \varphi$, is not compact

Symbols

Example

Non constant polynomials are symbols for $S(\mathbb{R})$. Also, if $P(x)$ is a polynomial with $\lim_{|x|\to+\infty} P(x) = +\infty$ then

$$\varphi(x) = \exp\left(P(x)\right)$$

is a symbol for $S(\mathbb{R})$.

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$C_\varphi : S(\mathbb{R}) \to S(\mathbb{R})$ is never supercyclic

Proof

(a) If $\varphi(x_1) = \varphi(x_2)$ for $x_1 \neq x_2$ then $\{\lambda C_{\varphi_n}(f) : \lambda \in \mathbb{C}, n \in \mathbb{N}\} \subseteq \mathrm{Ker}(\delta_{x_1} - \delta_{x_2})$.

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(b) If $\varphi(x_0) = x_0$ for some $x_0 \in \mathbb{R}$ then 1 is an eigenvalue and δ_{x_0} is an eigenvector from $C'_\varphi : S(\mathbb{R})' \to S(\mathbb{R})'$.

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(c) If φ is injective it should be surjective since it is a symbol. $S(\mathbb{R}) \hookrightarrow C_0(\mathbb{R})$ with dense range and C_φ is not supercyclic in $C_0(\mathbb{R})$ by Bourdon-Shapiro theorem, since it is an isometry.

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Fréchet case follows from Yosida theorem.

Dynamics

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(b) *Let φ be a decreasing symbol. Then C_φ is power bounded if and only if C_φ is mean ergodic if and only if $\varphi(\varphi(x)) = x$ for each $x \in \mathbb{R}$.*

- There are many decreasing symbols satisfying the condition in (b)
- We do not know if C_φ is mean ergodic for $\varphi(x) = x + e^{-x^2}$.

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We now consider $f \in \mathcal{S}(\mathbb{R})$, $f \equiv 1$ in $[-b, b]$. Since we are assuming that the composition operator is mean ergodic we can also consider

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Then

$$Pf(b) = \lim_k Pf(x_k) = 1.$$

On the other hand, $\lim_n f(\varphi_n(x)) = Pf(x)$ for all $x \in \mathbb{R}$ and every $n \in \mathbb{N}$ when the limit exists. Consequently, $Pf(x) = 0$ if the orbit $(\varphi_n(x))_n$ is unbounded, in particular $Pf(x) = 0$ when $|x| > b$, a contradiction.

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Spectra: Polynomials of degree 1

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or, equivalently, for $\psi := \varphi^{-1}$

$$f(x) = \lambda^n f(\psi_n(x)) + \sum_{k=1}^{n} \lambda^{k-1} g(\psi_k(x)).$$

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We take limits as $n \to \infty$ (with x fixed) in the two previous identities and obtain

$$f(x) = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{\lambda^k} g(\varphi_k(x))$$

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The contradiction comes easily from the discreteness of $(\varphi_n(0))_n$. There is g compactly supported such that $g(0) = 1$ and $g(\varphi_n(0)) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$.

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$$f = \lambda f \circ \psi + g \circ \psi$$

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$$\lim_k \left| \frac{f(y_k) - f(b)}{y_k - b} \right| = \infty.$$

Spectra: Polynomials of degree 2

Theorem

Let $\varphi(x) = a_0 + a_1 x + a_2 x^2$ *be a quadratic polynomial with real coefficients and take* $c = a_0 a_2 + \frac{a_1}{2} - \frac{a_1^2}{4}$.

(a) $c > \frac{1}{4}$ *implies* $\sigma\left(C_\varphi\right) = \{0\}$.

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The key point is the following fact, $c > 1/4$ means that there are not fixed points, $c = 1/4$ means that there is exactly one fixed point and $c < 1/4$ implies the existence of 2 fixed points.