Algebraic structures in linear dynamics

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Linear dynamics

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Linear dynamics studies the behaviour of orbits of (continuous, linear) operators

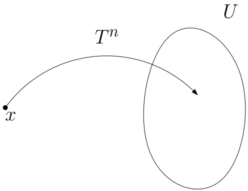
$$T : X \to X$$

on separable Banach or Fréchet spaces X.

The operator T is hypercyclic if there is a vector $x \in X$ such that

$$\mathrm{orb}(x, T) = \{x, Tx, T^2x, \ldots\}$$

is dense in X.



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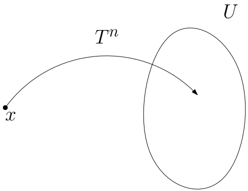

Each such vector x is then called a hypercyclic vector for T.
One denotes by

$$HC(T)$$

the set of all hypercyclic vectors.

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$\lambda B : (x_n) \to \lambda(x_{n+1})$, $|\lambda| > 1$: multiples of the backward shift

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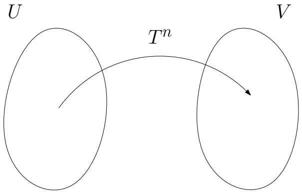
$$\Longleftrightarrow$$

for any non-empty open sets $U, V \subset X$ there is some $n \geq 0$ such that

$$T^n(U) \cap V \neq \varnothing.$$



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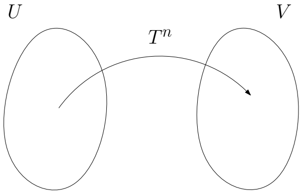

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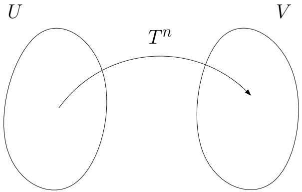

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The proof is by a simple application of Baire's theorem.

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It is also fundamentally non-linear: there is no good reason why the sum of two hypercyclic vectors should be hypercyclic. In fact, it follows from Birkhoff's theorem that

$$X = HC(T) + HC(T).$$

Nonethless, and perhaps surprisingly, something linear can be found in $HC(T)$:

Herrero-Bourdon-Bès-Wengenroth (1991–2003)
Let $T : X \to X$ be a hypercyclic operator.
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Proof: Take a hypercyclic vector x and look at

$$M = \text{span}\{x, Tx, T^2x, \ldots\}.$$

Lineability

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Definition

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The set $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is dense G_δ in $\mathbb{R}^2$ – and contains $\mathbb{R}(1, \sqrt{2})$, apart from $(0, 0)$.

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There are good necessary or sufficient conditions:
González-León-Montes (2000), Menet (2013-2014).

Theory of algebraic genericity

In recent years, motivated by the notions of lineability and spaceabilty, a general theory of algebraic genericity has been developped.

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There are two main sources of results in the theory:

- real and complex analysis
- linear dynamics

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Problem (Aron-Gurariy, 2003)

Is the subset of ℓ^∞ formed by the sequences that have only a finite number of zero coordinates spaceable in ℓ^∞?

(Problem 107 in Guirao, Montesinos, Zizler: Open problems in the geometry and analysis of Banach spaces, 2016)

Algebrability

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The research in this area was initiated by R. Aron (ca. 2005).

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So, given a hypercyclic operator T on a Banach or Fréchet algebra X, does T admit a hypercyclic algebra, in other words, is there a vector $x \in X$ so that any vector

$$\sum_{k=1}^{n} a_k x^k \neq 0, \quad n \geq 1,$$

is hypercyclic for T?

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Because, otherwise, there is some n such that

$$f^2(z + na) \approx z,$$

which is impossible by Hurwitz.

The same authors have a partial positive result for the differentiation operator D on $H(\mathbb{C})$.

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The positive answer was given independently in two publications.

Theorem (Bayart-Matheron 2009, Shkarin 2010)

$D : H(\mathbb{C}) \to H(\mathbb{C})$: *Yes.*

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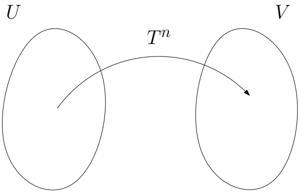
Recall Birkhoff:

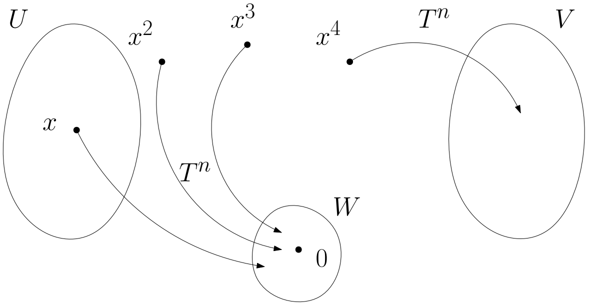


Lemma (Bayart-Matheron 2009)

Let $T : X \to X$ be an operator on a separable Fréchet algebra. If for any non-empty open sets $U, V \subset X$, any 0-neighbourhood $W \subset X$ and any $m \geq 1$ there is some $x \in U$ and some $n \geq 0$ such that

$$T^n(x^k) \in W \quad (1 \leq k < m) \quad \text{and} \quad T^n(x^m) \in V.$$

Then T admits a hypercyclic algebra.



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Proof: Let $(V_l)_l$ be a base of the topology of X and, for $l, r, m \in \mathbb{N}$,

$$E(l, r, m) = \{x \in X : \forall\, |a_1|, \ldots, |a_{m-1}| \leq r, a_m = 1 \\ \exists\, n \geq 0 : T^n\Big(\sum_{k=1}^{m} a_k x^k\Big) \in V_l\}.$$

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$$x \in \bigcap_{l,r,m} E(l,r,m) \Longrightarrow \forall\, m \,\forall\, a_1, \ldots a_m, \sum_{k=1}^{m} a_k x^k \in HC(T) \cup \{0\}.$$

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Aron 2000's

• Is $HC(D)$ even algebrable?

• Let $P \neq$ const be a polynomial. Does $P(D)$ admit a hypercyclic algebra? Is $HC(P(D))$ algebrable?

Theorem (Bès-Conejero-Papathanasiou 2017)

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Let $U, V \subset H(\mathbb{C})$ be non-empty open sets, and $m \geq 1$. Let $f \in U$, $g \in V$, which we may assume to be polynomials.

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Let $U, V \subset H(\mathbb{C})$ be non-empty open sets, and $m \geq 1$. Let $f \in U$, $g \in V$, which we may assume to be polynomials. Then one can find

$$f + h \in U,$$

where

$$h(z) = z^q \sum_{j=0}^{p} c_j z^j, \quad p = \deg(g),$$

is small and q and n and the c_j are chosen such that

$$(P(D))^n((f+h)^m) = (P(D))^n(h^m) = g$$

and $(P(D))^n((f+h)^k) = 0$ for $k = 1, \ldots, m-1$.

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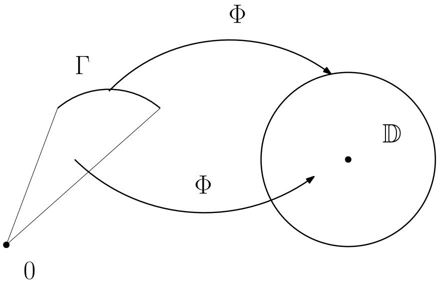
So, for which Φ does $\Phi(D)$ have a hypercyclic algebra?

Theorem (Bès-Conejero-Papathanasiou, JFA 2018)

Let $\Phi \neq$ const be an entire function of exponential type. Suppose that the level set $\{|\Phi(z)| = 1\}$ contains a non-trivial, strictly convex compact arc Γ so that

$$\mathit{conv}(\Gamma \cup \{0\}) \setminus (\Gamma \cup \{0\}) \subset \Phi^{-1}(\mathbb{D}).$$

Then $\Phi(D)$ admits a hypercyclic algebra.



The proof uses again the Bayart-Matheron lemma, but it replaces the action of $\Phi(D)$ on polynomials by the action on the eigenfunctions of D:

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$P(0) \neq 0$ possible:

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variants of e^D that do have hypercyclic algebras:
De^D, $e^D - I$

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Theorem (Bayart, arXiv 2018)

Let $\Phi \neq$ const be an entire function of exponential type with $|\Phi(0)| < 1$. Then $\Phi(D)$ admits a hypercyclic algebra if and only if Φ is not a multiple of an exponential function:

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He also obtains various sufficient conditions if $|\Phi(0)| = 1$.

In addition he shows that

$$\Phi(D) = 2e^{-D} + sin(D)$$

admits a hypercyclic algebra. Note that $\Phi(0) = 2$.

The only reasonable conjecture at this point seems to be:

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The operator $\Phi(D)$ admits a hypercyclic algebra if and only if Φ is not a multiple of an exponential function:

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But we do not even know if

$$\Phi(D) = 2I + D$$

admits a hypercyclic algebra.

In fact, we do not yet have good criteria for the absence of a hypercyclic algebra.

Problem

Find a general approach to show that an operator does not admit a hypercyclic algebra.

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Another problem:

Problem

Is there an operator T for which there is some $x \in X$ so that

$$x, x^2, x^3, \ldots$$

are hypercyclic but so that T does not admit any hypercyclic algebra?

Shift operators

Let's turn to the third classical example of a hypercyclic operator, the multiples of the backward shift:

$$T = \lambda B, |\lambda| > 1, \quad \text{on } X = \ell^p \text{ or } c_0.$$

Shift operators

Let's turn to the third classical example of a hypercyclic operator, the multiples of the backward shift:

$$T = \lambda B, |\lambda| > 1, \quad \text{on } X = \ell^p \text{ or } c_0.$$

Bayart and Matheron (2009) have shown (in an exercise) that T admits a hypercyclic algebra –
where X is endowed with the convolution product

$$(x_n) * (y_n) = \Big(\sum_{k=0}^{n} x_{n-k} y_k \Big)_n ,$$

which requires that $X = \ell^1$.

With J. Falcó we have looked at weighted shift operators:

$$B_w(x_0, x_1, x_2, \ldots) = (w_1 x_1, w_2 x_2, w_3 x_3, \ldots)$$

on general Banach or Fréchet algebras of sequences X; here

$$w = (w_n)$$

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is a sequence of non-zero scalars.
Also, X may carry either the coordinatewise product

$$(x_n).(y_n) = (x_n y_n)_n$$

or the convolution product

$$(x_n) * (y_n) = \Big(\sum_{k=0}^{n} x_{n-k} y_k\Big)_n.$$

One motivation is the fact that the differentiation operator

$$D : H(\mathbb{C}) \to H(\mathbb{C}), \quad f \to f'$$

turns into the weighted shift operator

$$(a_n) \to ((n+1)a_{n+1})_n,$$

when one identifies $H(\mathbb{C})$ with the sequence space

$$\Big\{(a_n) : \forall\, r > 0, \sum_{n=0}^{\infty} |a_n| r^n < \infty\Big\}$$

via the Taylor coefficients at 0.
Products in $H(\mathbb{C})$ correspond then to the convolution product of sequences.

Under the coordinatewise product one has a good control over orbits of any power x^k of any sequence x. Under slight technical assumptions on the space X and the weight w one obtains hypercyclic algebras. Our proof is constructive.

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In particular:

Theorem (Falcó-GE, preprint 2017)

Any hypercyclic weighted shift B_w on ℓ^p or c_0 or $H(\mathbb{C})$ (under the coordinatewise product) admits a hypercyclic algebra.

Of course, the set of scalars must be $\mathbb{C}$...

Of greater interest, and considerably more difficult, is the case of the convolution product. Again we obtain hypercyclic algebras under a technical assumption on the space X and if B_w is mixing, that is

$$\forall\ U, V \subset X \text{ non-empty open } \exists N\, \forall n \geq N : \quad T^n(U) \cap V \neq \varnothing.$$

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In particular:

Theorem (Falcó-GE, preprint 2017)

Any mixing weighted shift B_w on ℓ^1 or $H(\mathbb{C})$ (under the convolution product) admits a hypercyclic algebra.

This confirms that $\lambda B, |\lambda| > 1$, on ℓ^1 and D on $H(\mathbb{C})$ admit hypercyclic algebras.

The proof is based on the following idea.
In the proof of Bès-Conejero-Papathanasiou on $P(D)$ the polynomial $f \in U$ was perturbed by a (small) polynomial h such that

$$(P(D))^n((f+h)^m) = (P(D))^n(h^m) = g.$$

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We perturb a finite sequence $f \in U$ by a (small) finite sequence h and a (small) value b so that

$$B_w^n((f+h+be_M)^m) = B_w^n((mb^{m-1}e_{(m-1)M} * h + b^m e_{mM})) \approx g,$$

where $e_n = (0, \dots, 0, \underset{\uparrow n}{1}, 0, \dots)$

Aron’s question on algebrability

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In all the results stated or alluded to so far we have even the existence of an infinitely, not finitely generated algebra, that is, algebrability. In particular:

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The set $HC(D)$ of hypercyclic vectors for $D : H(\mathbb{C}) \to H(\mathbb{C})$ is algebrable.

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Theorem (Bès-Conejero-Papathanasiou, Falcó-GE 2017)

The set $HC(D)$ of hypercyclic vectors for $D : H(\mathbb{C}) \to H(\mathbb{C})$ is algebrable.

The following is open.

Question

Does there exist an operator for which $HC(T)$ contains an algebra but is not algebrable?

Multiplicative operators

What if the operator itself is multiplicative:

$$\forall\, x, y \in X, T(xy) = TxTy.$$

Does that help or hinder?

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Does that help or hinder?

For example, the translation operator

$$T_a : f \to f(\cdot + a)$$

is multiplicative on $H(\mathbb{C})$ – and it has no hypercyclic algebra.

But, in fact, we see that for a multiplicative operator T and any polynomial $P(t) = \sum_{k=1}^{p} a_k t^k$,

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Thus we have:

Theorem (Bès-Conejero-Papathanasiou, JFA 2018)

Let T be a hypercyclic multiplicative operator. Then T admits a hypercyclic algebra if and only if, for any polynomial P with $P(0) = 0$ the set

$$\{P(x) : x \in X\}$$

is dense in X.

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Theorem (Bès-Conejero-Papathanasiou, JFA 2018)

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is dense in X.

This is a property of the space alone.

Corollary (Bès-Conejero-Papathanasiou, JFA 2018)

The translation operator

$$T_a : f \to f(\cdot + a), a \neq 0$$

admits a hypercyclic algebra on the space $C^\infty(\mathbb{R}, \mathbb{C})$.

Corollary (Bès-Conejero-Papathanasiou, JFA 2018)

The translation operator

$$T_a : f \to f(\cdot + a), a \neq 0$$

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This is, of course, not true on $C^\infty(\mathbb{R}, \mathbb{R})$.

Closed subalgebras

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We finally mention the following (open) question.

Question (Shkarin 2010)

Does the operator D on $H(\mathbb{C})$ admit a closed hypercyclic algebra?

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We finally mention the following (open) question.

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A modification of the proof of the last mentioned result gives.

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The translation operator

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