Linear structures of continuous, integrable and unbounded functions

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6th July 2018

Previous Concepts

Definition

Let X be a topological vector space (t.v.s.), $A \subset X$. We say that

- *A is lineable if $\exists M \subset A \cup \{0\}$ v.s. of infinite dimension.*
- *A is dense-lineable if M can be chosen dense in X.*
- *A is maximal-(dense)-lineable if $dim(M) = dim(X)$.*

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Definition

Let X be contained in some (linear) algebra A and $\mathcal{B} \subset \mathcal{A}$. We say that

- *$\mathcal{B}$ is algebrable if $\exists \mathcal{C} \subset \mathcal{A}$ so that $\mathcal{C} \subset \mathcal{B} \cup \{0\}$ and the cardinality of any system of generators of $\mathcal{C}$ is infinite.*
- *If in addition, $\mathcal{A}$ is a commutative algebra, we say that $\mathcal{B}$ is strongly algebrable if $\mathcal{B} \cup \{0\}$ contains generated algebra which is isomorphic to a free algebra.*

Known Results

1. Gurariy, 1966: $\aleph_0$-lineability of Weierstrass' Monsters.
2. Fonf, Gurariy, Kadets, 1999: Spaceability of Weiertrass' Monsters.
3. Jiménez-Rodríguez, Muñoz-Fernández, Seoane-Sepúlveda, 2013: $\mathfrak{c}$-lineability of Weierstrass' Monsters.
4. Albuquerque, 2014: Maximal-lineability of the set of continuous surjections from $\mathbb{R}$ to $\mathbb{R}^2$.
5. Muñoz, Palmberg, Puglisi, Seoane: $\mathfrak{c}$-lineability of $L^p[0,1]\setminus L^q[0,1]$ for $1 \leq p < q$.
6. García, Martín, Seoane, 2009: $\mathfrak{c}$-lineability of the set of Lebesgue integrable functions that are no Riemann integrable.
7. Lineability of $\mathcal{DNM}(\mathbb{R})$ in $\mathcal{C}(\mathbb{R})$, ...

Lineability

Example

Consider the triangular function $T_n : [0, +\infty) \longrightarrow \mathbb{R}$ *given by:*

$$T_n(x) = \begin{cases} n(2^{n+1}x + (1 - n2^{n+1})) & \text{if } x \in [n - 1/2^{n+1}, n), \\ n(-2^{n+1}x + (1 + n2^{n+1})) & \text{if } x \in [n, n + 1/2^{n+1}], \\ 0 & \text{otherwise.} \end{cases}$$

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and the function $f : [0, +\infty) \longrightarrow \mathbb{R}$ *defined by the previous triangles:*

$$f(x) = \sum_{n=1}^{+\infty} T_n(x).$$

Lineability

Theorem

The family $\mathcal{A}$ of unbounded continuous integrable functions, that is, the set

$$\mathcal{A} = \left\{ f \in \mathcal{C}([0,+\infty)) \cap L^1([0,+\infty)) \ : \ \limsup_{x \to +\infty} |f(x)| = +\infty \right\}$$

is maximal lineable.

Lineability

Lemma

Let X be a metrizable topological vector space, $A \subset X$ maximal lineable and $B \subset X$ dense-lineable in X with $A \cap B = \varnothing$. If A is stronger than B then A is maximal dense-lineable.

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We define in $X = \mathcal{C}([0,+\infty)) \cap L^1([0,+\infty))$ the metric

$$d_X(f,g) = \|f-g\|_{L^1} + \sum_{n=1}^{+\infty} \frac{1}{2^n} \cdot \frac{\|f-g\|_{\infty,[0,n]}}{1+\|f-g\|_{\infty,[0,n]}}.$$

Lineability

Consider B the family of all the functions

$$b_{n,\gamma}(x) = \begin{cases} p(x) & \text{if} \quad 0 \leq x \leq n, \\ \dfrac{p(n)}{\gamma}(n - x + \gamma) & \text{if} \quad n < x \leq n + \gamma, \\ 0 & \text{if} \quad x > n + \gamma, \end{cases}$$

where $p(x)$ is a polygonal, $n \in \mathbb{N}$ and $\gamma > 0$.

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The family $\mathcal{A}$ of unbounded continuous integrable functions maximal dense-lineable.

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Consider the "triangles" given by:

$$T_{n,p}(x) = \begin{cases} n^p(2^{n+1}x + (1 - n2^{n+1}))^p & \text{if } x \in [n - 1/2^{n+1}, n), \\ n^p(-2^{n+1}x + (1 + n2^{n+1}))^p & \text{if } x \in [n, n + 1/2^{n+1}], \\ 0 & \text{otherwise}, \end{cases}$$

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and we define the functions $g_p : [0, +\infty) \longrightarrow \mathbb{R}$ *as:*

$$g_p(x) = \sum_{n=1}^{+\infty} T_{n,p}(x).$$

Algebrability

Theorem

The family $\mathcal{A}$ of unbounded continuous integrable functions is strongly-algebrable.

Known Results

1. Araújo, Bernal, Muñoz, Prado and Seoane, 2017: $\mathfrak{c}$-lineability of sequences in $\mathcal{MES}$.
2. Araújo, Bernal, Muñoz, Prado and Seoane, 2017: Maximal dense-lineability of sequences in $L_0([0,1])$ such that $f_n \to 0$ in measure but not pointwise a.e..

Lineability

$$\mathcal{A}_0 := \{(f_n)_n \ : \ f_n \in \mathcal{A}, \in \mathbb{N}, f_n \to 0 \text{ pointwise on } [0, +\infty), \|f_n\|_{L^1} \to 0\}.$$

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Consider the sequence of functions $(f_n)_n$ given by:

$$f_n(x) := \sum_{m=n}^{+\infty} T_m(x), \quad \forall n \in \mathbb{N}, x \geq 0.$$

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Theorem

The family of sequences $\mathcal{A}_0$ on $[0,+\infty)$ is maximal lineable.

Lineability

Consider now the spaces

$$c_0(X) := \{F = (f_n)_n : f_n \in X\ n \in \mathbb{N}, d_X(f_n, 0) \to 0\},$$

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Algebrability

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Final Remarks

- $\mathcal{A}$: function f continuous integrable unbounded $\Longrightarrow \begin{cases} \text{1. Maximal dense-lineability.} \\ \text{2. Strong-algebrability.} \end{cases}$

- $\mathcal{A}_0$: sequence $(f_n)_n$ continuous integrable unbounded almost unif. $\Longrightarrow \begin{cases} \text{1. Maximal dense-lineability.} \\ \text{2. Strong-algebrability.} \end{cases}$

Thank you very much for your attention

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