On the Fourier spectrum of functions on Boolean cubes

Joint work of: Andreas Defant, Mieczysław Mastyło, and Antonio Pérez

Paweł Domański Memorial Conference – 1-7 July 2018

The Bohnenblust-Hille inequality on the polytorus

Point of departure:

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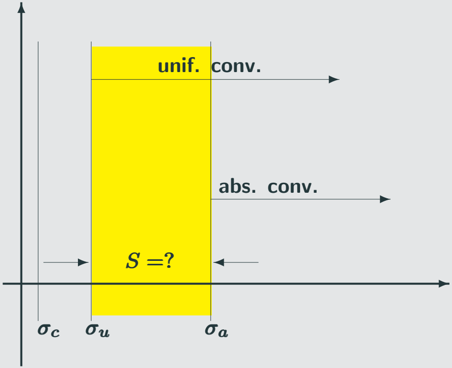
Bohr's absolute convergence problem for Dirichlet series $D = \sum a_n n^{-s}$ from 1914



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  all $f \in L_\infty(\mathbb{T}^\infty)$ such that $\hat{f}(\alpha) = 0$ whenever $\alpha < 0$

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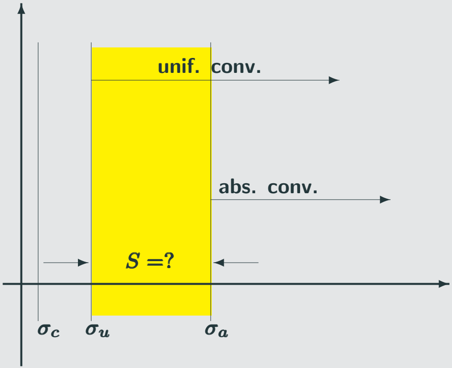
Come back



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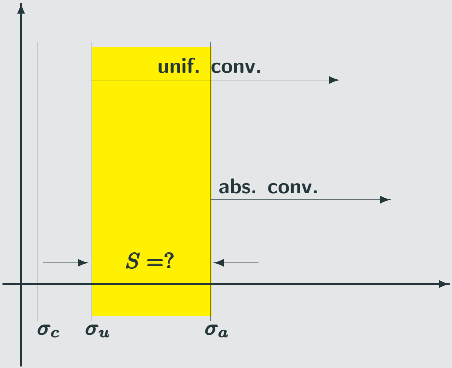
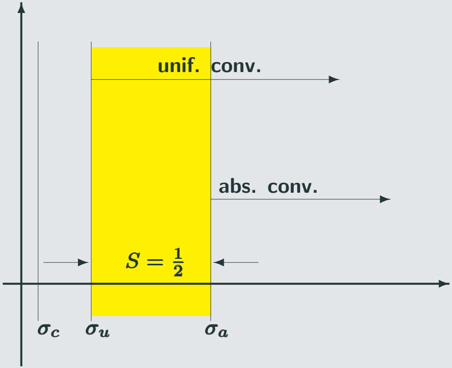


Reformulation of Bohr's problem

$$S = \sup_{D \in \mathcal{H}_\infty} \sigma_a(D)$$

Bohr-Bohnenblust-Hille Theorem, 1931



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Bohr, Bohnenblust-Hille, Queffélec, Queffélec-Balasubramanian, Queffélec-Konyagin, de la Bretèche, Defant-Frerick-Seip-Ortega....

Driving force – Bohnenblust-Hille-inequality, 1931

For each $d \in \mathbb{N}$ there is a (best) constant $\mathrm{BH}_{\mathbb{T}}^{\leq d}$ such that every degree-d polynomial $f = \sum_{|\alpha| \leq d} c_\alpha z^\alpha$ on $\mathbb{C}^N$

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Bayart-Pellegrino-Seoane 2014:

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Defant-Mastyło-Pérez 2018

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Compare

$$\overline{\lim}_d \sqrt[d]{\mathrm{BH}^{\leq d}_{\mathbb{T}}} = \overline{\lim}_d \sqrt[d]{\mathrm{BH}^{=d}_{\mathbb{T}}} = 1$$

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The BH-inequality on the Boolean cube

Analysis of functions on the Boolean cube

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Example – majority function

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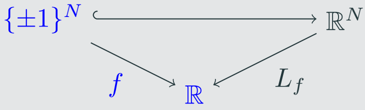
d-homogeneous functions: $\hat{f}(S) \neq 0$ only if $|S| = d$

Another point of view – tetrahedral polynomials



$$\sup_{x \in \{\pm 1\}^N} |f(x)| = \sup_{x \in [-1,1]^N} |L_f(x)|$$

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Again......

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Again – majority function

$$\mathsf{Maj}(x) = \mathsf{sign}(x_1 + \ldots + x_N)$$

Finding Fourier coefficients of functions on the Boolean cube may not be easy:

For N odd

$$\widehat{\mathsf{Maj}}(S) = \begin{cases} 0 & |S| \quad \text{even} \\ (-1)^{\frac{|S|-1}{2}} \frac{1}{2^{N-1}} \binom{N-1}{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{\frac{|S|-1}{2}} \binom{N-1}{|S|-1}^{-1} & |S| \quad \text{odd} \end{cases}$$

Recall again

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Moreover, the exponent is optimal.

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For each $d \in \mathbb{N}$ there is a (best) constant $\mathrm{BH}^{\leq d}_{\{\pm 1\}}$ such that for every $f : \{\pm 1\}^N \to \mathbb{R}$ of degree d

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Moreover, the exponent is optimal.

Blei's constant is big!

$$\mathrm{BH}^{\leq d}_{\{\pm 1\}} \ll (d+1)^{\frac{d+1}{2d}} (d!)^{\frac{d-1}{2d}} \sqrt{d}^{d+1} (2e)^d$$

Problem, Montanaro 2013

$$\exists\, C \geq 1\ \forall\, d : \quad \mathrm{BH}^{\leq d}_{\{\pm 1\}} \leq d^C\ ?$$

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Main result, Defant-Mastyło-Pérez 2018

There exists a constant $C > 0$ such that for all d

$$\mathrm{BH}^{\leq d}_{\{\pm 1\}} \leq C^{\sqrt{d \log d}}$$

In particular,

$$\overline{\lim}_d \sqrt[d]{\mathrm{BH}^{\leq d}_{\{\pm 1\}}} = 1\,.$$

Why could it be interesting to improve the constants?

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In the article 'The need for structure in quantum speedups' by Aaronson and Ambainis from 2011 the following informal problem is studied:

For every 'unstructured' problem, are the quantum complexity and the classical (randomized) complexity polynomially related ?

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.........Conjecture 4 – foklore

Every quantum query algorithm can be approximated by a classical algorithm on 'most' inputs.

This follows from the so-called AA-conjecture which in fact is a conjecture in Fourier analysis on the Boolean cube....

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For every $f : \{\pm 1\}^N \to [-1, 1]$ of degree d, is there some $j \in [N]$ such that

$$\left(\frac{\sum_{S \neq \emptyset} \widehat{f}(S)^2}{d}\right)^{O(1)} \leq \sum_{S : j \in S} \widehat{f}(S)^2 \quad \textbf{?}$$

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O'Donnell: Analysis of Boolean functions

If true, this conjecture would have significant consequences regarding the limitations of efficient quantum computation.

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- $\mathrm{BH}_{\{\pm 1\}}^{\leq d} \leq d^{O(1)} \quad \Rightarrow \quad$ AA true for $f : \{\pm 1\}^N \to \{\pm \alpha\}$
  Montanaro 2015

Surprising or not surprising?

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Recall

- The real BH-inequality: $\overline{\lim}_d \sqrt[d]{\mathrm{BH}^{\leq d}_{[-1,1]}} = 1 + \sqrt{2}$
- The complex BH-inequality: $\overline{\lim}_d \sqrt[d]{\mathrm{BH}^{\leq d}_{\mathbb{T}}} = 1$

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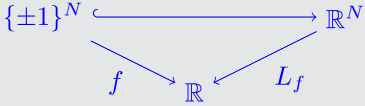
- The real BH-inequality: $\overline{\lim}_d \sqrt[d]{\mathrm{BH}^{\leq d}_{[-1,1]}} = 1 + \sqrt{2}$
- ....and the tetrahedral case is somewhat in between!
- The complex BH-inequality: $\overline{\lim}_d \sqrt[d]{\mathrm{BH}^{\leq d}_{\mathbb{T}}} = 1$



$$\sup_{x \in \{\pm 1\}^N} |f(x)| = \sup_{x \in [-1,1]^N} |L_f(x)|$$

How to prove this?

Littlewood's $4/3$-inequality, 1931

For every degree-2 polynomial $P(z) = \sum_{|\alpha|\leq 2} c_\alpha z^\alpha$ on $\mathbb{C}^N$

$$\Big(\sum_{|\alpha|\leq 2} |c_\alpha|^{\frac{4}{3}} \Big)^{\frac{3}{4}} \leq \mathrm{BH}_{\mathbb{T}}^{\leq 2} \, \|P\|_{\mathbb{T}^N}$$

Concentrate on the homogeneous case....

For every 2-homogeneous polynomial $P(z) = \sum_{1 \leq i \leq j \leq N} c_{ij} z_i z_j$ on $\mathbb{C}^N$

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How to prove this?

- There are four steps,
- and every step seems interesting in itself....

Step 1 – symmetric forms

$$P(z) = \sum_{1 \leq i \leq j \leq N} c_{ij} z_i z_j \quad \longleftrightarrow \quad L_P(u,v) = \sum_{1 \leq i,j \leq N} a_{ij} u_i v_j \ ,$$

where $c_{ii} = a_{ii}$, and $c_{ij} = 2a_{ij}$ for $i < j$.

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Step 2 – mixed terms

$$\Big(\sum_{1 \le i,j \le N} |a_{ij}|^{\frac{4}{3}} \Big)^{\frac{3}{4}} \le \Big[\sum_i \Big(\sum_j |a_{ij}|^2 \Big)^{\frac{1}{2}} \times \sum_j \Big(\sum_i |a_{ij}|^2 \Big)^{\frac{1}{2}} \Big]^{\frac{1}{2}}$$

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Step 3 – Kinchine's inequality

$$\Big(\sum_j |\lambda_j|^2\Big)^{\frac{1}{2}} \le \sqrt{2}\int_{\mathbb{T}^N} \Big|\sum_i \lambda_j z_i\Big| dz$$

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$$\|L_P\|_{\mathbb{T}^N \times \mathbb{T}^N} \leq 2\|P\|_{\mathbb{T}^N}$$

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Alltogether

$$\Big(\sum_{1 \leq i \leq j \leq N} |c_{ij}|^{\frac{4}{3}} \Big)^{\frac{3}{4}} \ll \Big(\sum_{1 \leq i \leq j \leq N} |a_{ij}|^{\frac{4}{3}} \Big)^{\frac{3}{4}} \ll \|L_P\|_{\mathbb{T}^N \times \mathbb{T}^N} \ll \|P\|_{\mathbb{T}^N}$$

How to extend these steps in order to prove our main result?

$$\Big(\sum_{|S| \leq d} |\widehat{f}(S)|^{\frac{2d}{d+1}} \Big)^{\frac{d+1}{2d}} \leq C^{\sqrt{d \log d}} \|f\|_{\{\pm 1\}^N}$$

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The final outcome is the following estimate which allows iteration....

$$\mathrm{BH}^{\leq d}_{\{\pm 1\}} \ \leq \ e^{\left(4\sqrt{d \log d}\right)} \ \ \mathrm{BH}^{\leq \lfloor \sqrt{d/\log d} \rfloor}_{\{\pm 1\}}$$

Step 1 – fully decoupled polynomials on $\mathbb{R}^n$ (Kwapien 1989)

For every polynomial $P : \mathbb{R}^N \longrightarrow \mathbb{R}$ of degree d there is a unique mapping $L_P : (\mathbb{R}^N)^d \longrightarrow \mathbb{R}$ satisfying the following properties:

- L_P is separately affine
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This 'associated d-affine form' is given through the following polarization formula:

$$L_P(x^{(1)}, \ldots, x^{(d)}) \\ = \mathbb{E}_\xi \left[\sum_{m=0}^{d} \frac{\xi_1 \ldots \xi_d}{d!} (\xi_1 + \ldots + \xi_n)^{d-m} P_m(\xi_1 x^{(1)} + \ldots + \xi_d x^{(d)}) \right]$$

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Compare....

Notation

$$\mathcal{M}(d,N) = [N]^d \text{ and } \mathcal{M}(S,N) = S^d \text{ whenever } S \subset [d]$$

Step 2 – mixed terms (Bayart-Pellegrino-Seoane 2014)

$$\left(\sum_{\mathbf{i} \in \mathcal{M}(d,N)} |a_{\mathbf{i}}|^{\frac{2d}{d+1}} \right)^{\frac{d+1}{2d}}$$

$$\leq \left[\prod_{\substack{S \subset [d] \\ |S|=k}} \left(\sum_{\mathbf{i}_1 \in \mathcal{M}(S,N)} \left(\sum_{\mathbf{i}_2 \in \mathcal{M}(\hat{S},N)} |a_{\mathbf{i}_1 \oplus \mathbf{i}_2}|^2 \right)^{\frac{1}{2} \frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \right]^{\frac{1}{\binom{d}{k}}}$$

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Compare....

Step 3 – hypercontractivity of the noise operator (Bonami-Gross 1970/75)

For $-1 < \rho < 1$ the noise operator is defined by

$$f(x) = \sum_{S \subset [N]} \widehat{f}(S) x^S \ \mapsto \ T_\rho f(x) = \sum_{S \subset [N]} \widehat{f}(S) \rho^{|S|} x^S$$

Then for any $1 < p \leq q \leq \infty$ and $\rho \leq \sqrt{\frac{p-1}{q-1}}$

$$\Big\| T_\rho : L_p\big(\{\pm 1\}^N\big) \longrightarrow L_q\big(\{\pm 1\}^N\big) \Big\| \leq 1 \,.$$

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Consequence – Kinchine type inequalities à la:

For every degree-d function $f : \{\pm1\}^N \to \mathbb{R}$

$$\|f\|_2 \leq e^d\|f\|_1 \quad \text{and} \quad \|f\|_2 \leq \big(1/\sqrt{p-1}\big)\|f\|_p\,,\ 1 < p \leq 2\,.$$

Harris' polarization estimate, 1998

Let $P : \mathbb{C}^N \longrightarrow \mathbb{C}$ be a d-homogeneous polynomial of degree d and L_P its associated symmetric d-linear form. Then, for each $0 \leq m \leq d$ we have that

$$\sup_{x,y \in \mathbb{T}^N} \left| L_P(\underbrace{x, \ldots, x}_{m}, \underbrace{y, \ldots, y}_{d-m}) \right| \leq \frac{m!}{m^m} \frac{(d-m)!}{(d-m)^{d-m}} \frac{d^d}{d!} \sup_{x \in \mathbb{T}^N} |P(x)| .$$

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Step 4 – polarization (Defant-Mastyło-Pérez 2018)

Let $P : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a polynomial of degree d and L_P its associated d-affine form. Then, for each $0 \leq m \leq d/2$ we have that

$$\sup_{x,y \in [-1,1]^N} \left| L_P(\underbrace{x, \dots, x}_{m}, \underbrace{y, \dots, y}_{d-m}) \right| \leq 2d^m \sup_{x \in [-1,1]^N} |P(x)|.$$

A combination of all steps finally leads to

$$\mathrm{BH}_{\{\pm 1\}}^{\leq d} \leq \mathrm{BH}_{\{\pm 1\}}^{\leq \lfloor \sqrt{d/\log d} \rfloor} \exp\left(4\sqrt{d \log d}\right),$$

and then by iteration as desired to

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More applications: Bohr's phenomenon for functions on the Boolean cube, Defant-Mastyło-Pérez, JFA 2018

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Still open:

$$\mathbf{BH}^{\leq d}_{\{\pm 1\}} \leq d^{C}\ ?$$