The best dominated approximation in the sense of the Hardy-Littlewood-Pólya relation in symmetric spaces.

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PAWEŁ DOMAŃSKI MEMORIAL CONFERENCE
1-7 July 2018, Będlewo, Poland

GOAL

1. Relationship between **strict K-monotonicity** and **uniqueness** of the best dominated approximation problem with respect to the **Hardy-Littlewood-Pólya relation** $\prec$ in **symmetric spaces**.
2. Essential question whether **proximinality** of the best dominated approximation problem with respect to the **Hardy-Littlewood-Pólya** relation $\prec$ corresponds to K**-order continuity** in symmetric spaces.
3. Characterization of K**-order continuity** in symmetric spaces. Complete criteria for K**-order continuity** using a notion of the best dominated approximation with respect to $\prec$.

[1] M. Ciesielski, *Hardy-Littlewood-Pólya relation in the best dominated approximation in symmetric spaces*, J. Approx. Theory **213** (2017), 78-91.
[2] M. Ciesielski, *Relationships between K-monotonicity and rotundity properties with application*, J.Math.Anal.Appl. **465** (2018), no. 1, 235-258

Introduction

Let (Ω, Σ, μ) be a complete σ-finite measure space and $L^0(\Omega)$ be a space of all classes of $f : \Omega \to \overline{\mathbb{R}}$ μ-**measurable** extended real-valued functions.

(Quasi-) Banach ideal space E, is a linear subspace of $L^0(\Omega)$, equipped with a complete quasi-norm $\|\cdot\|_E : E \to \mathbb{R}_+$ satisfying:

(*i*) If $f \in L^0(\Omega)$, $g \in E$ and $|f| \leq |g|$ a.e., then $f \in E$, $\|f\|_E \leq \|g\|_E$,

(*ii*) There exists a **weak unit**, i.e. strictly positive $f \in E$.

If (Ω, Σ, μ) is non-atomic then E is said to be **(quasi-) Banach function space**.

We assume that E has **Fatou property**, i.e. for any $(x_n) \subset E^+$,

$$\sup_{n\in\mathbb{N}} \|x_n\|_E < \infty \quad \text{and} \quad x_n \uparrow x \in L^0(\Omega) \quad \Rightarrow \quad x \in E \quad \text{and} \quad \|x_n\|_E \uparrow \|x\|_E .$$

Introduction

The **distribution function** of $x \in L^0(\Omega)$

$$d_x(\lambda) = \mu\left(\{t \in \Omega : |x(t)| > \lambda\}\right) \qquad \text{for all} \quad \lambda \geq 0.$$

E is called **symmetric** or **rearrangement invariant** (r.i.) whenever $x \in L^0(\Omega)$, $y \in E$ and $d_x = d_y$ we have $x \in E$, $\|x\|_E = \|y\|_E$.

Consider $I = [0, \alpha)$, where $\alpha = \infty$ or $\alpha = 1$ and μ is the **Lebesgue measure** on $\mathbb{R}$. The **decreasing rearrangement** of $x \in L^0(I)$

$$x^*(t) = \inf\left\{s > 0 : d_x(s) \leq t\right\}, \text{ for } t > 0.$$

The **maximal function** of x^*

$$x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)ds, \text{ for } t > 0.$$

The **Hardy-Littlewood-Pólya relation** $\prec$ is given for any $x, y \in L^1 + L^\infty$ by

$$x \prec y \Leftrightarrow x^{**}(t) \leq y^{**}(t) \text{ for all } t > 0.$$

Consider $\mathcal{A} \subset E$ and $x \in E$. Denote

$$\mathcal{A} \prec x \quad (\text{resp. } x \prec \mathcal{A}) \quad \text{if for all} \quad y \in \mathcal{A}, \quad y \prec x \quad (\text{resp. } x \prec y).$$

Introduction

A function $x \in L^0(I)$ is said to be ***regular** if

$$\mu(t \in I : |x(t)| < x^*(\alpha)) = 0,$$

where

$$x^*(\alpha) = \lim_{t \to \infty} x^*(t) \quad \text{if} \quad \alpha = \infty \qquad \text{and} \qquad x^*(\alpha) = 0 \quad \text{if} \quad \alpha = 1.$$

Let $Y \subset X$, $Y \neq \emptyset$. For $x \in X$ define

$$P_Y(x) = \{y \in Y : \|x - y\| = dist(x, Y)\}.$$

Any element $y \in P_Y(x)$ is called a **best approximant** in Y to x.

A nonempty set $Y \subset X$ is called **proximinal** or **set of existence** if $P_Y(x) \neq \emptyset$ for any $x \in X$.

A nonempty set Y is said to be a **Chebyshev set** if $P_Y(x)$ is a singleton for any $x \in X$.

Introduction

A point $x \in E$ is a **point of upper K-monotonicity** (**a point of lower K-monotonicity**) shortly **a** *UKM* **point** (**an** *LKM* **point**) of E if for any $y \in E$,

$$x^* \neq y^*, \qquad x \prec y \quad (y \prec x) \quad \Rightarrow \quad \|x\|_E < \|y\|_E \quad (\|y\|_E < \|x\|_E),$$

respectively.

A symmetric space E is said to be **strictly K-monotone** ($E \in (SKM)$) if any $x \in E$ is a *UKM* point (or equivalently x is an *LKM* point).

A point $x \in E$ is a **point of order continuity** ($x \in E_a$) if for any $(x_n) \subset E$,

$$0 \leq x_n \leq |x| \quad \text{and} \quad x_n \to 0 \quad \text{a.e.} \quad \Rightarrow \quad \|x_n\|_E \to 0.$$

E is called **order continuous** ($E \in (OC)$) if and only if $E = E_a$.

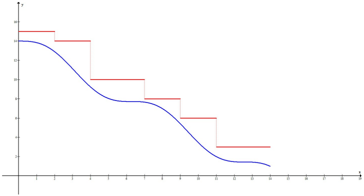
A point $x \in E$ is a **point of K-order continuity** of E if for any $(x_n) \subset E$,

$$x_n \prec x \quad \text{and} \quad x_n^* \to 0 \quad \text{a.e.} \quad \Rightarrow \quad \|x_n\|_E \to 0.$$

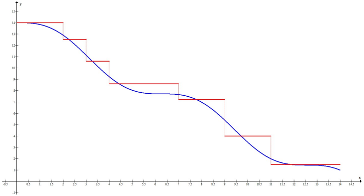
A symmetric space E is called K-**order continuous**, shortly $E \in (KOC)$, if every $x \in E$ is a point of K-order continuity.

What's the difference between the best d. approximation with respect to $\prec$ and $\leq$

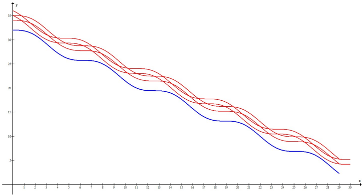
$f \leq g$



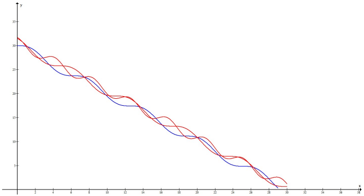
$f \prec g$



$f \leq g_n$

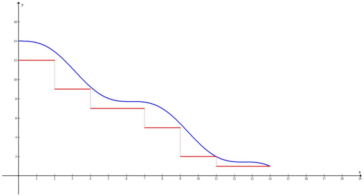


$f \prec g_n$

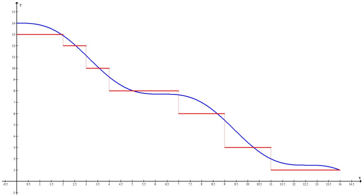


What's the difference between the best d. approximation with respect to $\prec$ and $\leq$

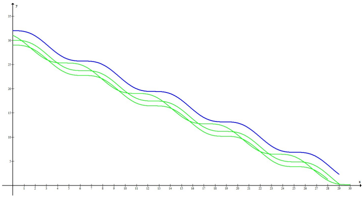
$g \leq f$



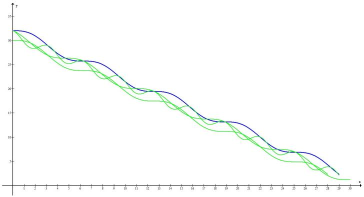
$g \prec f$



$h_n \leq f$



$h_n \prec f$



Uniqueness of the best dominated approximation problem with respect to $\prec$

Theorem (2.1)

Let E be a symmetric space and $x \in E^+$. If for any closed subset $K \subset E$ with

$$x \prec K$$

we have $$card(P_K(x)) \leq 1,$$
*then x is a UKM **point** and an LKM **point**.*

Theorem (2.2)

Let E be a symmetric space and $x \in E^+$ be *regular. If for any closed subset $K \subset E$ with

$$K \prec x$$

we have $$card(P_K(x)) \leq 1,$$
*then x is an LKM **point**.*
*Additionally, if $y^*(\infty) = 0$ for all $y \in E$, then x is a UKM **point**.*

Uniqueness of the best dominated approximation problem with respect to $\prec$

Corollary (2.3)

Let E be a symmetric space. If for any closed subset $K \subset E$ and $x \in E$ with

$$x \prec K,$$

we have

$$card(P_K(|x|)) \leq 1,$$

*then E is **strictly K-monotone**.*

Corollary (2.4)

Let E be a symmetric space such that $x^*(\infty) = 0$ for any $x \in E$. If for any closed subset $K \subset E$ and $x \in E$ with

$$K \prec x$$

we have

$$card(P_K(|x|)) \leq 1,$$

*then E is **strictly K-monotone**.*

Introduction to Lorentz spaces $\Gamma_{p,\omega}$ and $\Lambda_{p,\omega}$

Let $0 < p < \infty$ and $w \in L^0$ be a nonnegative weight function. The **Lorentz space** $\Lambda_{p,w}$ $\Gamma_{p,w}$ is a subspace of L^0 such that

$$\|f\|_{\Lambda_{p,w}} = \left(\int_0^\alpha f^{*p}(t)w(t)dt\right)^{1/p} < \infty,$$

$$\|f\|_{\Gamma_{p,w}} = \|f^{**}\|_{\Lambda_{p,w}} = \left(\int_0^\alpha f^{**p}(t)w(t)dt\right)^{1/p} < \infty,$$

respectively.

Lorentz spaces are interpolation spaces between L^1 and L^∞.
$\Lambda_{p,w}$ and $\Gamma_{p,w}$ are r.i. quasi-Banach function spaces with the Fatou property, where for $\Lambda_{p,w}$ the mapping $W(t) = \int_0^t w$ is positive and satisfies Δ_2-condition on I.

Introduction to Lorentz spaces $\Gamma_{p,\omega}$ and $\Lambda_{p,\omega}$

Additional assumptions

$$W(x) = \int_0^x w < \infty, \tag{1}$$

$$W_p(x) = x^p \int_x^\alpha t^{-p} w(t)dt < \infty, \tag{2}$$

$$W(\infty) = \infty, \tag{3}$$

for all $0 < x < \alpha$.

If conditions (1) and (2) hold then $w \in D_p$ and $\Gamma_{p,w} \neq \{0\}$.

If $\alpha = \infty$ then condition (3) is equivalent to fact that $\Lambda_{p,w}$ and $\Gamma_{p,w}$ are **order continuous**.

Introduction - History of Lorentz spaces $\Gamma_{p,\omega}$ and $\Lambda_{p,\omega}$

- Lorentz, 1950,
  Introduction to $\Lambda_{p,w}$, *Some new functional spaces*.
- Calderón, 60-ties,
  The Lorentz space $\Gamma_{p,w}$ appeared explicitly.
- Hunt, 60-ties,
  $\Lambda_{p,w} = \Gamma_{p,w}$, where $w(t) = t^{p/q-1}$, $1 < p, q < \infty$.
- Sawyer, 90-ties,
  Köthe dual of $\Lambda_{p,w}$ coincides with $\Gamma_{p',\widetilde{w}}$ whenever $1 < p < \infty$ and $W(\infty) = \infty$, where $p' = \frac{p}{p-1}$ and $\widetilde{w}(x) = \left(\frac{x}{W(x)}\right)^{p'} w(x)$.
- Kamińska & Maligranda, 2004,
  $\Lambda_{p,w}$ and $\Gamma_{p,w}$ have **order continuous** norm and contains an **order isomorphic** and **complement copy** of l^p under some additional assumption on w. Conditions for p**-convexity** and q**-concavity**.

Uniqueness of the best dominated approximation problem with respect to $\prec$

Example (2.5)

Consider $\Gamma_{p,w}$ with $w > 0$, $\int_0^\infty w < \infty$ and $p > 0$. Clearly, $\Gamma_{p,w}$ is SKM (see Theorem 2.10 [1] *). Define*

$$A = \bigcup_{n=0}^{\infty}[2n, 2n+1), \qquad x = \left(1 + \frac{1}{1+t}\right)\chi_I,$$

$$u = 2\chi_I, \qquad v = u + \frac{1}{1+t}\chi_A, \qquad \text{and} \qquad \mathcal{K} = [u, v].$$

By Proposition 2.1 [2]*, we have $x \in \Gamma_{p,w}$ and $\mathcal{K} \subset \Gamma_{p,w}$. Moreover,*

$$x \prec \mathcal{K} \quad \text{and} \quad (x - z)^* = 1 \quad \text{for any } z \in \mathcal{K},$$

whence $P_{\mathcal{K}}(x) = \mathcal{K}$ is not a singleton.

[1] M. Ciesielski, P. Kolwicz and R. Płuciennik, *Note on strict K-monotonicity of some symmetric function spaces*, Comm. Math. 53 (2) (2013), 311-322.

[2] M. Ciesielski, A. Kamińska, P. Kolwicz and R. Płuciennik, *Monotonicity and rotundity of Lorentz spaces $\Gamma_{p,w}$*, Nonlinear Anal. 75 (2012), 2713-2723.

History of K-order continuity

- P.G. Dodds, E.M. Semenov and F.A. Sukochev, 2004,
  Introduce a new notion in E a separable symmetric space X with the Fatou property,
  $$(DSS) \qquad \text{If} \quad x_n \prec x \quad \text{and} \quad x_n \to 0 \quad \text{in measure} \quad \Rightarrow \quad \|x_n\|_X \to 0,$$
  for any $(x_n) \subset X$, $x \in X$
- P.G. Dodds, T.K. Dodds, and F.A. Sukochev, 2007,
  Investigate (DSS) in the more general setting of symmetric spaces of measurable operators under additional assumption.
- M.C., 2017,
  Complete criteria of K-order continuity in symmetric spaces with application to the best dominated approximation problem with respect to the Hardy-Littlewood-Pólya relation.
- M.C. and G. Lewicki, 2017,
  Relationship between KOC and rotundity and monotonicity properties with application to the best dominated approximation problem with respect to $\prec$ on the directed set.

K-order continuity in symmetric spaces

Lemma (3.1)

Let E be a symmetric space with the fundamental function ϕ and let $(x_n) \subset E, x \in E$. If $y^*(\infty) = 0$ for any $y \in E$, then F.C.A.E.

- (*i*) *If $x_n \prec x$ and $x_n^* \to 0$ a.e., then $\|x_n\|_E \to 0$, (E is KOC).*
- (*ii*) *If $x_n \prec x$ and $x_n \to 0$ globally in measure, then $\|x_n\|_E \to 0$, (E is DSS).*

Theorem (3.2)

Let E be a symmetric space on $I = [0, 1)$. E is KOC if and only if E is OC.

Theorem (3.3)

Let E be a symmetric space with the fundamental function ϕ on $I = [0, \infty)$. F.C.A.E.

- (*i*) *E is order continuous and is not embedded in $L^1[0, \infty)$.*
- (*ii*) *E is order continuous and $E' \hookrightarrow \{f : f^*(\infty) = 0\}$.*
- (*iii*) *E is K-**order continuous** and $\phi(\infty) = \infty$.*

K-order continuity in the Orlicz spaces

The mapping $\psi : \mathbb{R} \to [0, \infty]$ is called an **Orlicz function** if ψ is nonzero, even, convex, continuous and vanishes at zero and

$$\lim_{|t|\to\infty} \psi(t) = \infty.$$

The Orlicz function ψ satisfies Δ_2 **condition** for all $u \in \mathbb{R}^+$, shortly $\psi \in \Delta_2$, if there exists $K > 0$ such that for all $u \in \mathbb{R}$ we have $\psi(2u) \leq K\psi(u)$. The **Orlicz space** L^ψ is a subspace of L^0 such that

$$\rho_\psi(\lambda x) = \int_I \psi(\lambda x(t))dt < \infty, \qquad \text{for some } \lambda > 0.$$

It is well known that the Orlicz space L^ψ might be considered as a Banach space equipped with the **Luxemburg norm**

$$\|x\|_\psi = \inf\left\{\lambda > 0 : \rho_\psi\left(\frac{x}{\lambda}\right) \leq 1\right\}$$

or with the equivalent **Orlicz norm**

$$\|x\|_\psi^o = \sup\left\{\left|\int_I y(t)x(t)dt\right| : \rho_{\psi_Y}(y) \leq 1\right\}.$$

K-order continuity in the Orlicz spaces

Theorem (3.4)

Let ψ be an Orlicz function and Φ_{L^ψ} be the fundamental function of the Orlicz space L^ψ. Then, we have

(1) *The Orlicz space L^ψ on $I = [0, 1)$ is **K-order continuous** if and only if*

$$\psi \in \Delta_2.$$

(2) *The Orlicz space L^ψ on $I = [0, \infty)$ is **K-order continuous** and $\Phi_{L^\psi}(\infty) = \infty$ if and only if*

$$\psi \in \Delta_2 \quad \text{and} \quad \lim_{t \to 0} \frac{\psi(t)}{t} = 0.$$

Proximinality of the best dominated approximation problemwith respect to $\prec$, $(\mathcal{A} \prec x)$

Theorem (3.5)

Let E be a symmetric space and let $\mathcal{A} \subset E$ be a closed subset such that

$$a^* \in \mathcal{A} \quad \text{for any} \quad a \in \mathcal{A}.$$

*If $x \in E$ is a **point of K-order continuity** such that*

$$x^*(\infty) = 0 \quad \text{and} \quad \mathcal{A} \prec x,$$

then the set $P_{\mathcal{A}}(x^*) \neq \emptyset$.

Theorem (3.6)

Let E be a symmetric space and $x \in E^+$ be *regular. If for any closed subset $\mathcal{A} \subset E$ such that $\mathcal{A} \prec x$ we have

$$P_{\mathcal{A}}(x) \neq \emptyset,$$

*then x is a **point of K-order continuity**.*

Proximinality of the best dominated approximation problem with respect to $\prec$, $(\mathcal{A} \prec x)$

Corollary (3.7)

Let E be a symmetric space with $x^*(\infty) = 0$ for any $x \in E$ and let $\mathcal{A} \subset E$ be a closed subset such that

$$a^* \in \mathcal{A} \quad \text{for any} \quad a \in \mathcal{A}.$$

*If E is K**-order continuous** and*

$$x \in E \quad \text{with} \quad \mathcal{A} \prec x,$$

then the set $P_{\mathcal{A}}(x^*) \neq \emptyset$.

Corollary (3.8)

Let E be a symmetric space with $x^*(\infty) = 0$ for any $x \in E$. If for any closed subset $\mathcal{A} \subset E$ and for any $x \in E$ with $\mathcal{A} \prec x$ we have

$$P_{\mathcal{A}}(x) \neq \emptyset,$$

*then E is K**-order continuous**.*

Proximinality of the best dominated approximation problem with respect to $\prec$, $(x \prec \mathcal{A})$

A point $a \in E$ is called a K**-upper bound** of a subset $\mathcal{A} \subset E$ if for any $a' \in \mathcal{A}$ we have $a' \prec a$.

If there exists a K-upper bound of a subset $\mathcal{A} \subset E$, then the set $\mathcal{A}$ is said to be K**-bounded above**.

Proximinality of the best dominated approximation problem with respect to $\prec$, $(x \prec \mathcal{A})$

Theorem (3.9)

Let E be a symmetric space and $x \in E$. If for any closed K-bounded above subset $\mathcal{A}$ of E such that

$$x \prec \mathcal{A} \quad \text{and} \quad a^* \in \mathcal{A} \quad \text{for any } a \in \mathcal{A}$$

we have

$$P_{\mathcal{A}}(x^*) \neq \emptyset,$$

*then x is a **point of K-order continuity**.*

Theorem (3.10)

Let E be a symmetric space, $x \in E$ and let $\mathcal{A} \subset E$ be a closed K-bounded above subset such that

$$x \prec \mathcal{A}, \quad a^* \in \mathcal{A} \quad \text{for any } a \in \mathcal{A}.$$

*If a K-upper bound of $\mathcal{A}$ is a **point of K-order continuity** with finite distribution, then we have*

$$P_{\mathcal{A}}(x^*) \neq \emptyset.$$

Proximinality of the best dominated approximation problem with respect to $\prec$, $(x \prec \mathcal{A})$

Theorem (3.11)

Let E be a symmetric space with $x^*(\infty) = 0$ for any $x \in E$. The F.C.A.E.

*(i) E is K***-order continuous***.*

(ii) For any $x \in E$ and $\mathcal{A} \subset E$ a closed K-bounded above subset such that

$$x \prec \mathcal{A}, \quad a^* \in \mathcal{A} \quad \text{for any } a \in \mathcal{A},$$

we have $P_{\mathcal{A}}(x^*) \neq \emptyset$.

Thank You for your attention.

C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics Series 129, Academic Press Inc.,1988.

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M. Ciesielski, P. Kolwicz and A. Panfil, *Local monotonicity structure of symmetric spaces with applications*, J. Math. Anal. Appl. 409 (2014) 649-662.

M. Ciesielski, P. Kolwicz and R. Płuciennik, *Local approach to Kadec-Klee properties in symmetric function spaces*, J. Math. Anal. Appl. 426 (2015) 700-726.

M. Ciesielski, P. Kolwicz and R. Płuciennik, *Note on strict K-monotonicity of some symmetric function spaces*, Comm. Math. 53 (2) (2013), 311-322.

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H. Hudzik and A. Kamińska, *Monotonicity properties of Lorentz spaces*, Proc. Amer. Math. Soc. **123.9**, (1995), 2715-2721.