ON THE Ext^2 PROBLEM IN BANACH SPACES

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1. Banach space theory is about spaces $X, Y, \ldots$ and operators $T, S, \ldots$ acting between them.

If $\mathfrak{L}(X,Y)$ denotes the Banach space of linear continuous operators acting between X and Y, one can say that at the ground level Banach space theory treats about the functor $\mathfrak{L}$.

2. To consider subspaces and quotients and therefore be able to treat extension and lifting results the suitable object is that of exact sequence

$$0 \longrightarrow Y \xrightarrow{\ j\ } Z \xrightarrow{\ q\ } X \longrightarrow 0$$

whose meaning is:

- Y is a subspace of Z through the isomorphic embedding j
- and X is the corresponding quotient $Z/j(Y)$.

Or else

- $q : Z \to X$ is a quotient map
- and $Y = \ker q$.

The exact sequence splits if Y is complemented in Z.

3. Other operative questions require to step one level up. That next level is formed by the spaces $\mathrm{Ext}(X,Y)$ of exact sequences as above (modulo equivalence). Thus, those more elaborate questions about the behaviour of spaces and operators are questions about the functor Ext.

For instance:

- Sobczyk's theorem can be formulated as $\mathrm{Ext}(S, c_0) = 0$ for every separable Banach space S;
- Lindenstrauss's lifting principle can be formulated as $\mathrm{Ext}(\mathcal{L}_1, U) = 0$ for every ultrasummand U;
- Johnson-Zippin theorem can be formulated as $\mathrm{Ext}(H^*, \mathcal{L}_\infty) = 0$ for every subspace H of c_0.

4. The functors Ext and $\mathfrak{L}$ are connected: Ext is the derived functor of $\mathfrak{L}$.

It is thus natural that questions about the variation of operators are actually questions about Ext.

5. It was Kalton's discovery that exact sequences of quasi-Banach spaces can be described by using a certain type of nonlinear maps, called quasi-linear maps.

Therefore, the space $\mathrm{Ext}(X,Y)$ corresponds to a space $\mathfrak{Q}(X,Y)$ of functions

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] \arrow[ll, bend right] & 0
\end{tikzcd}$$

modulo equivalence.

6. It is natural then that other operative questions require to step one level up.

And these questions can easily be formulated now using quasi-linear maps: questions such as when quasi-linear maps on a given subspace extend to a given superspace, for instance.

7. That next level is formed by the spaces $\mathrm{Ext}^2(X,Y)$, where Ext^2 is the derived functor of Ext and second derived functor of $\mathfrak{L}$. It is so that questions about when $\mathrm{Ext}^2(X,Y)=0$ emerge.

8. To fix ideas now we can admit that the elements of Ext^2 are concatenations

$$GF$$

of quasi-linear maps

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & Y' \arrow[r] & \diamondsuit \arrow[r] \arrow[ll, bend right, "G"'] & X' \arrow[r] & X \arrow[r] \arrow[ll, bend right, "F"'] & 0
\end{tikzcd}$$

modulo some equivalence.

9. An element GF

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & Y' \arrow[r] & \diamondsuit \arrow[r] \arrow[ll, bend right, "G"'] & X' \arrow[r] & X \arrow[r] \arrow[ll, bend right, "F"'] & 0
\end{tikzcd}$$

splits when it .... splits.....

$$\begin{tikzcd}
Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & \\
Y' \arrow[r] \arrow[d] & \square \arrow[r] \arrow[d] & X \arrow[d, equal] \\
\diamondsuit \arrow[r] \arrow[uu, bend left=60, "G"] & X' \arrow[r] & X \arrow[ll, bend left, "F"]
\end{tikzcd}$$

10. Thus, the immediate meaning of a question

$$\mathrm{Ext}^2(X,Y)=0$$

is whether extension and lifting theorems hold for quasi-linear maps.

11. There are however more meanings.

Questions about Ext can be formulated as questions about operators on subspaces of ℓ_1 or operator into quotients of ℓ_∞.

For instance

$$\mathrm{Ext}(\ell_2, \ell_2) = 0$$

is the question of whether every operator $D_2 \to \ell_2$ defined on the kernel $D_2 = \ker q$ of a quotient map $q : \ell_1 \to \ell_2$ can be extended to an operator $\ell_1 \to \ell_2$ (is 2-summing).

The answer is no and is provided by the celebrated Kalton-Peck twisted Hilbert space

$$0 \longrightarrow \ell_2 \longrightarrow Z_2 \longrightarrow \ell_2 \longrightarrow 0$$

12. Questions about Ext^2 are deeper questions about subspaces of ℓ_1 or quotients of ℓ_∞. Many questions are simply unknown for those spaces:

- What the kernel of a quotient map $\ell_1 \to \ell_2$ looks like?
- What is like $\ell_\infty / C[0,1]$? Is it a $C(K)$-space?

The first question is connected with the Ext^2 problem for Hilbert spaces: Is

$$\mathrm{Ext}^2(\ell_2, \ell_2) = 0?$$

The second question is connected to the Ext^2 problem for $C[0,1]$.

13. Other questions in these lines are simply intractable. Such are the questions about the homological dimension of a Banach space. For instance: Does this stops:

$$\cdots \longrightarrow \kappa(\kappa(X)) \longrightarrow \ell_1 \longrightarrow \kappa(X) \longrightarrow \ell_1 \longrightarrow X \longrightarrow 0$$

Is there a natural n so that $\kappa(\kappa(\cdots(X)\cdots)) = \ell_1$?

Bourgain's celebrated space provides the only known example $B^* \neq \ell_1$ of a space with homological dimension 1, namely $k(B^*) = \ell_1$.

14. The Ext^2 problem for Hilbert spaces:

$$\text{Is} \quad \mathrm{Ext}^2(\ell_2, \ell_2) = 0?$$

is actually a question about the behavior of bilinear forms (instead of operators).

15. The Ext^2 problem for Hilbert spaces contains questions about the existence of twisted sums of twisted Hilbert spaces; in particular about the existence of twisted sums of the celebrated Kalton-peck Z_2-space. For instance, this occurs

$$\begin{tikzcd}
\ell_2 \arrow[r, equal] \arrow[d] & \ell_2 \arrow[d] & \\
Z_2 \arrow[r] \arrow[d] & \square \arrow[r] \arrow[d] & \ell_2 \arrow[d, equal] \\
\ell_2 \arrow[r] \arrow[uu, bend left, "K"] & Z_2 \arrow[r] & \ell_2 \arrow[ll, bend left, "K"]
\end{tikzcd}$$

if and only if $KK = 0$.

16. It is at this point that complex interpolation theory enters the game to establish that whenever a quasi-linear map Ω has been obtained as a differential in a complex interpolation schema then $\Omega\Omega = 0$ in Ext^2. Why?

(1)

$$\begin{tikzcd}
\ker\delta_{z_0}\cap\ker\delta'_{z_0} \arrow[rr, equal, dashed] \arrow[dr] \arrow[ddd, "{\delta_{z_0}}''"'] & & \ker\delta_{z_0}\cap\ker\delta'_{z_0} \arrow[dr] \arrow[ddd, dotted, "{\delta_{z_0}}''"] & & & & \\
& \ker\delta_{z_0} \arrow[rr] \arrow[dr, "{\delta_{z_0}}'"] \arrow[ddd] & & \mathcal{F} \arrow[rr, "\delta_{z_0}"] \arrow[dr] \arrow[ddd] & & \ell_2 \arrow[dr, equal, dashed] \arrow[ddd, equal, dashed] & \\
& & \ell_2 \arrow[rr] \arrow[d, equal, dashed] & & Z_2 \arrow[rr] \arrow[ddd, equal] & & \ell_2 \arrow[ddd, equal] \\
\ell_2 \arrow[rr, equal, dashed] \arrow[dr] & & \ell_2 \arrow[dr] \arrow[dd, equal, dashed] & & & & \\
& Z_2 \arrow[rr] \arrow[dr] & & \square\square\square\square \arrow[rr] \arrow[dr] & & \ell_2 \arrow[dr, equal, dashed] & \\
& & \ell_2 \arrow[rr] & & Z_2 \arrow[rr] & & \ell_2
\end{tikzcd}$$

17. The question of whether $\Omega\Omega = 0$ connects with commutator estimates. For instance, the Kalton-Peck map is $Kx = x\log|x|$. It is a miracle that this map can be extended.

$$\begin{tikzcd}
\ell_2 \arrow[r, equal] \arrow[d] & \ell_2 \arrow[d] & \\
Z_2 \arrow[r] \arrow[d] & \square \arrow[r] \arrow[d] & \ell_2 \arrow[d, equal] \\
\ell_2 \arrow[uu, bend left, "K"] \arrow[r] & Z_2 \arrow[r] & \ell_2 \arrow[ll, bend left, "K"]
\end{tikzcd}$$

18. The question of whether $\Omega\Lambda = 0$ is deeper. It means if we can have a diagram

$$\begin{tikzcd}
X_\theta \arrow[r, equal] \arrow[d] & X_\theta \arrow[d] & \\
d_\Omega X_\theta \arrow[r] \arrow[d] & \square \arrow[r] \arrow[d] & X_\theta \arrow[d, equal] \\
X_\theta \arrow[uu, bend left, "\Omega"] \arrow[r] & d_\Lambda X_\theta \arrow[r] & X_\theta \arrow[ll, bend left, "\Lambda"]
\end{tikzcd}$$

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