

## ON THE $\text{Ext}^2$ PROBLEM IN BANACH SPACES

FÉLIX CABELLO SÁNCHEZ, JESÚS M. F. CASTILLO, AND RICARDO GARCÍA

1. Banach space theory is about spaces  $X, Y, \dots$  and operators  $T, S, \dots$  acting between them.

If  $\mathfrak{L}(X, Y)$  denotes the Banach space of linear continuous operators acting between  $X$  and  $Y$ , one can say that at the ground level Banach space theory treats about the functor  $\mathfrak{L}$ .

2. To consider subspaces and quotients and therefore be able to treat extension and lifting results the suitable object is that of exact sequence

$$0 \longrightarrow Y \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0$$

whose meaning is:

- $Y$  is a subspace of  $Z$  through the isomorphic embedding  $j$
- and  $X$  is the corresponding quotient  $Z/j(Y)$ .

Or else

- $q : Z \rightarrow X$  is a quotient map
- and  $Y = \ker q$ .

The exact sequence splits if  $Y$  is complemented in  $Z$ .

3. Other operative questions require to step one level up. That next level is formed by the spaces  $\text{Ext}(X, Y)$  of exact sequences as above (modulo equivalence). Thus, those more elaborate questions about the behaviour of spaces and operators are questions about the functor  $\text{Ext}$ .

For instance:

- Sobczyk's theorem can be formulated as  $\text{Ext}(S, c_0) = 0$  for every separable Banach space  $S$ ;
- Lindenstrauss's lifting principle can be formulated as  $\text{Ext}(\mathcal{L}_1, U) = 0$  for every ultrasummand  $U$ ;
- Johnson-Zippin theorem can be formulated as  $\text{Ext}(H^*, \mathcal{L}_\infty) = 0$  for every subspace  $H$  of  $c_0$ .

4. The functors  $\text{Ext}$  and  $\mathfrak{L}$  are connected:  $\text{Ext}$  is the derived functor of  $\mathfrak{L}$ .

It is thus natural that questions about the variation of operators are actually questions about  $\text{Ext}$ .

5. It was Kalton's discovery that exact sequences of quasi-Banach spaces can be described by using a certain type of nonlinear maps, called quasi-linear maps.

Therefore, the space  $\text{Ext}(X, Y)$  corresponds to a space  $\mathfrak{Q}(X, Y)$  of functions

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

modulo equivalence.

6. It is natural then that other operative questions require to step one level up.

And these questions can easily be formulated now using quasi-linear maps: questions such as when quasi-linear maps on a given subspace extend to a given superspace, for instance.

7. That next level is formed by the spaces  $\text{Ext}^2(X, Y)$ , where  $\text{Ext}^2$  is the derived functor of  $\text{Ext}$  and second derived functor of  $\mathfrak{L}$ . It is so that questions about when  $\text{Ext}^2(X, Y) = 0$  emerge.

8. To fix ideas now we can admit that the elements of  $\text{Ext}^2$  are concatenations

$$GF$$

of quasi-linear maps

$$0 \longrightarrow Y \longrightarrow Y' \longrightarrow \diamond \longrightarrow X' \longrightarrow X \longrightarrow 0$$

$\overset{G}{\curvearrowright}$                        $\overset{F}{\curvearrowright}$

modulo some equivalence.

9. An element  $GF$

$$0 \longrightarrow Y \longrightarrow Y' \longrightarrow \diamond \longrightarrow X' \longrightarrow X \longrightarrow 0$$

$\overset{G}{\curvearrowright}$                        $\overset{F}{\curvearrowright}$

splits when it .... splits.....

$$\begin{array}{ccccccc}
 & & Y & \equiv & Y & & \\
 & & \downarrow & & \downarrow & & \\
 & & Y' & \longrightarrow & \square & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \parallel \\
 & & \diamond & \longrightarrow & X' & \longrightarrow & X \\
 & & \uparrow & & \uparrow & & \\
 & & G & & F & & 
 \end{array}$$

10. Thus, the immediate meaning of a question

$$\text{Ext}^2(X, Y) = 0$$

is whether extension and lifting theorems hold for quasi-linear maps.

11. There are however more meanings.

Questions about  $\text{Ext}$  can be formulated as questions about operators on subspaces of  $\ell_1$  or operator into quotients of  $\ell_\infty$ .

For instance

$$\text{Ext}(\ell_2, \ell_2) = 0$$

is the question of whether every operator  $D_2 \rightarrow \ell_2$  defined on the kernel  $D_2 = \ker q$  of a quotient map  $q : \ell_1 \rightarrow \ell_2$  can be extended to an operator  $\ell_1 \rightarrow \ell_2$  (is 2-summing).

The answer is no and is provided by the celebrated Kalton-Peck twisted Hilbert space

$$0 \longrightarrow \ell_2 \longrightarrow Z_2 \longrightarrow \ell_2 \longrightarrow 0$$

12. Questions about  $\text{Ext}^2$  are deeper questions about subspaces of  $\ell_1$  or quotients of  $\ell_\infty$ . Many questions are simply unknown for those spaces:

- What the kernel of a quotient map  $\ell_1 \rightarrow \ell_2$  looks like?
- What is like  $\ell_\infty/C[0,1]$ ? Is it a  $C(K)$ -space?

The first question is connected with the  $\text{Ext}^2$  problem for Hilbert spaces: Is

$$\text{Ext}^2(\ell_2, \ell_2) = 0?$$

The second question is connected to the  $\text{Ext}^2$  problem for  $C[0,1]$ .

13. Other questions in these lines are simply intractable. Such are the questions about the homological dimension of a Banach space. For instance: Does this stops:

$$\cdots \longrightarrow \kappa(\kappa(X)) \longrightarrow \ell_1 \longrightarrow \kappa(X) \longrightarrow \ell_1 \longrightarrow X \longrightarrow 0$$

Is there a natural  $n$  so that  $\kappa(\kappa(\cdots(X)\cdots)) = \ell_1$ ?

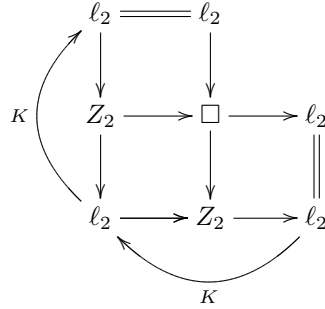
Bourgain's celebrated space provides the only known example  $B^* \neq \ell_1$  of a space with homological dimension 1, namely  $k(B^*) = \ell_1$ .

14. The  $\text{Ext}^2$  problem for Hilbert spaces:

$$\text{Is } \text{Ext}^2(\ell_2, \ell_2) = 0?$$

is actually a question about the behavior of bilinear forms (instead of operators).

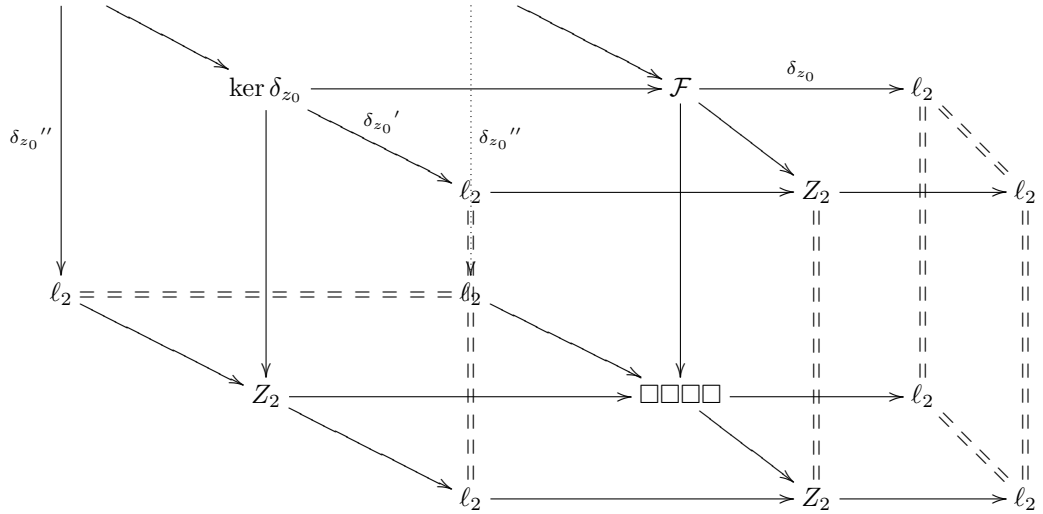
15. The  $\text{Ext}^2$  problem for Hilbert spaces contains questions about the existence of twisted sums of twisted Hilbert spaces; in particular about the existence of twisted sums of the celebrated Kalton-peck  $Z_2$ -space. For instance, this occurs



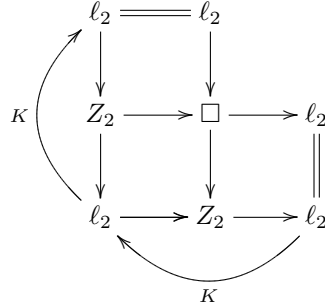
if and only if  $KK = 0$ .

16. It is at this point that complex interpolation theory enters the game to establish that whenever a quasi-linear map  $\Omega$  has been obtained as a differential in a complex interpolation schema then  $\Omega\Omega = 0$  in  $\text{Ext}^2$ . Why?

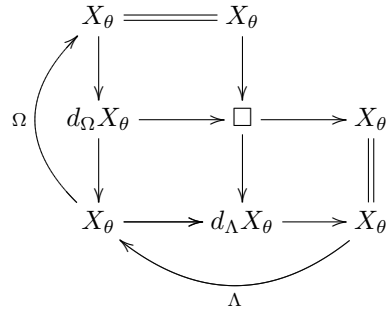
(1)  $\ker \delta_{z_0} \cap \ker \delta'_{z_0} = \dots = \ker \delta_{z_0} \cap \ker \delta''_{z_0}$



17. The question of whether  $\Omega\Omega = 0$  connects with commutator estimates. For instance, the Kalton-Peck map is  $Kx = x \log|x|$ . It is a miracle that this map can be extended.



18. The question of whether  $\Omega\Lambda = 0$  is deeper. It means if we can have a diagram



## REFERENCES

- [1] F. Cabello Sánchez, J.M.F. Castillo, *The long homology sequence for quasi-Banach spaces, with applications*, Positivity 8 (2004) 379-394.
- [2] F. Cabello, Jesús M. F. Castillo, *Stability constants and the homology of quasi-Banach spaces*, Israel J. Math. 198 (2013) 347-370.
- [3] F. Cabello, Jesús M. F. Castillo, Willian Correa, *Kalton vs. Rochberg derivation of analytic families of Banach spaces*, preprint.
- [4] F. Cabello, Jesús M. F. Castillo, N. J. Kalton. *Complex interpolation and twisted Hilbert spaces*, Pacific J. Math. (2015) 276 (2015) 287 - 307.
- [5] F. Cabello, Jesús M. F. Castillo, S. Goldstein, Jesús Suárez, *Twisting noncommutative  $L_p$ -spaces*, Advances in Math. 294 (2016) 454—488.
- [6] Jesús M.F. Castillo, Willian Correa, V. Ferenczi, M. González. *On the stability of the differential process generated by complex interpolation*, preprint
- [7] Jesús M. F. Castillo, R García, *Bilinear forms in the homology of Banach spaces*, preprint.
- [8] N.J. Kalton, *Differentials of complex interpolation processes for Köthe function spaces*, Trans. Amer. Math. Soc. 333 (1992) 479–529.
- [9] N.J. Kalton, N. T. Peck, *Twisted sums of sequence spaces and the three space problem*, Trans. Amer. Math. Soc. 255 (1979) 1-30.
- [10] V. Palamodov, *The projective limit functor in the category of topological linear spaces*. (Russian) Mat. Sb. (N.S.) 75 (117) 1968 567–603 (English Transl. Math-USSR-Sb 4 (1968) 529-558.

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE EXTREMADURA, AVENIDA DE ELVAS, 06071-BADAJOS, SPAIN  
*E-mail address:* `fcabello@unex.es`

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE EXTREMADURA, AVENIDA DE ELVAS, 06071-BADAJOS, SPAIN  
*E-mail address:* `castillo@unex.es`

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE EXTREMADURA, AVENIDA DE ELVAS, 06071-BADAJOS, SPAIN  
*E-mail address:* `rgarcia@unex.es`