

A separable Fréchet space of almost universal disposition

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joint work with Jerzy Kąkol and Wiesław Kubiś

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for all $x \in E$ is called an **ε -isometric embedding**.



Theorem (Banach-Mazur, 1929)

The space $(C[0, 1], \|\cdot\|_\infty)$ is universal for all separable Banach spaces.

In other words, for every separable Banach space E there is an isometric embedding $E \hookrightarrow C[0, 1]$.



The Gurariĭ space

In 1965, V. I. Gurariĭ constructed a separable Banach space with the following extension property.

- (G) For every $\varepsilon > 0$, for all finite dimensional normed spaces $E \subseteq F$, for every isometric embedding $e: E \rightarrow \mathbb{G}$ there exists an ε -isometric embedding $f: F \rightarrow \mathbb{G}$ such that $f \upharpoonright E = e$.



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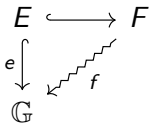
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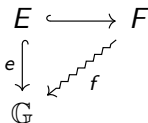
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In 1976, W. Lusky showed that (G) defines \mathbb{G} uniquely up to isometry. A simpler proof: Kubiś and Solecki (2013)



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If in addition,

$$\| \cdot \|_1 \leq \| \cdot \|_2 \leq \dots$$

we call E a **graded Fréchet space**.



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Theorem (Mazur-Orlicz, 1948)

The space $(\mathcal{C}(\mathbb{R}), \{\|\cdot\|_i\}_{i \in \mathbb{N}})$, where

$$\|f\|_i := \sup \{|f(x)| : x \in [-i, i]\},$$

is universal for all separable Fréchet spaces.

In other words, for every separable Fréchet space E there is an isometric embedding $E \hookrightarrow \mathcal{C}(\mathbb{R})$.



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A natural candidate is $\mathbb{G}^{\mathbb{N}}$. Can we find a suitable sequence of semi-norms on $\mathbb{G}^{\mathbb{N}}$?



Connecting to the Banach space setting

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$$E \hookrightarrow \prod_i E_i \quad \text{and} \quad E_{i+1} \twoheadrightarrow E_i$$

- If $f: E \rightarrow F$ is (ε) -isometric, so is $f_i: E_i \rightarrow F_i$, defined by

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \text{can} \downarrow & & \downarrow \text{can} \\ E_i & \xrightarrow{f_i} & F_i \end{array}$$



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Based on a result by Cabello Sánchez, Garbulińska-Węgrzyn, Kubiś (2014).



A graded sequence of semi-norms

We construct inductively a sequence of semi-norms on $\mathbb{G}^{\mathbb{N}}$.
For $x = (x_1, x_2, \dots) \in \mathbb{G}^{\mathbb{N}}$ we define

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$$\mathbb{G} = (\text{im } \pi) \oplus (\text{ker } \pi) \simeq \mathbb{G} \oplus \mathbb{G} = \mathbb{G} \times \mathbb{G}$$

holds isometrically and that there is a norm $\|\cdot\|'_2$ on \mathbb{G}^2 such that \mathbb{G}^2 is isometric to \mathbb{G} and

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Define $\|x\|_2 := \|(x_1, x_2)\|'_2$ and inductively $\|\cdot\|_n$ in a similar way.



Proposition

The space $(\mathbb{G}^{\mathbb{N}}, \{\|\cdot\|_i\}_{i \in \mathbb{N}})$ is a graded Fréchet space of almost universal disposition for finite dimensional graded Fréchet spaces, i.e., for all $\varepsilon > 0$ and for all finite dimensional graded Fréchet spaces $E \subseteq F$ and all isometric embeddings $f: E \rightarrow \mathbb{G}^{\mathbb{N}}$ there is an ε -isometric embedding $g: F \rightarrow \mathbb{G}^{\mathbb{N}}$ such that $g \upharpoonright E = f$.



Given $\varepsilon > 0$, finite dimensional graded Fréchet spaces E and F and an isomeric embedding $f: E \rightarrow F$, choose a sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ with $\varepsilon_i > 0$ and

$$\prod_{i=1}^{\infty} (1 + \varepsilon_i) < 1 + \varepsilon.$$

Set

$$E_i = (E / \ker \|\cdot\|_i, \|\cdot\|_i) \quad \text{and} \quad F_i = (F / \ker \|\cdot\|_i, \|\cdot\|_i)$$

and $f_i: E_i \rightarrow \mathbb{G}^i$ defined by

$$\begin{array}{ccc} E & \xrightarrow{f} & \mathbb{G} \\ \text{can} \downarrow & & \downarrow \text{can} \\ E_i & \xrightarrow{f_i} & \mathbb{G}^i. \end{array}$$



Proof sketch II: Defining g inductively

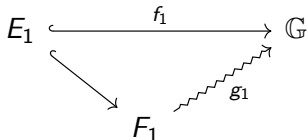
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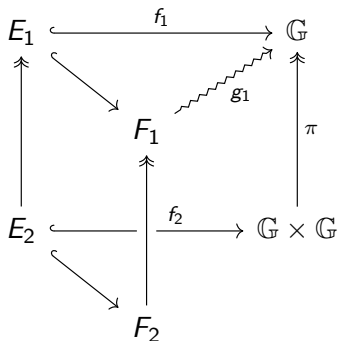


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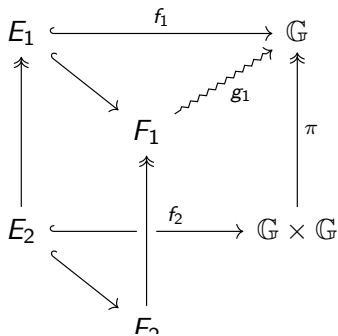


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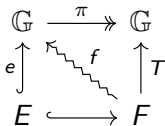


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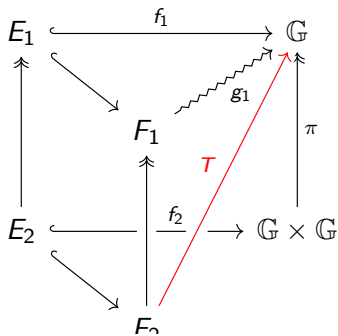
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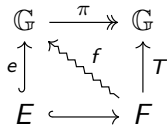


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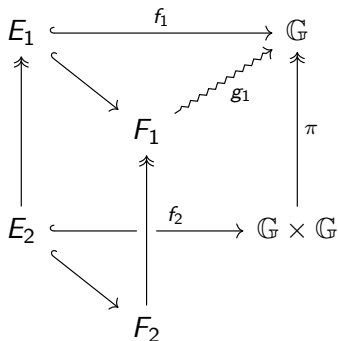
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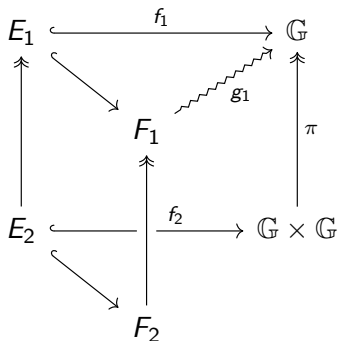


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g_1 is an ε_1 -isometry. Since $\|g_1 \circ p_1^2\| > 1$ is possible, we need an additional step.



Proof sketch III: The additional step

Lemma

Let $X \subset Y$ and A be finite dim. Banach spaces, Z a Banach space, $e: X \hookrightarrow A$, $T: Y \rightarrow Z$ with $\|T\| < r$, $r > 1$, and

$\pi: A \rightarrow Z$, $\|\pi\| \leq 1$, s.t.

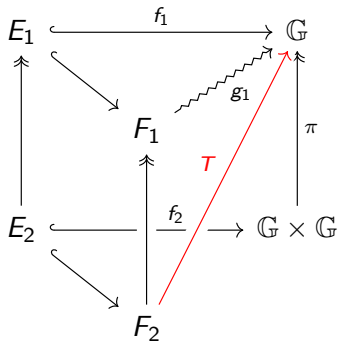
$$\begin{array}{ccc} A & \xrightarrow{\pi} & Z \\ \uparrow e & & \uparrow T \\ X & \hookrightarrow & Y \end{array}$$

There exists a finite dim. Banach space C , $i_A: A \hookrightarrow C$, an $(r-1)$ -isometric embedding $i_Y: Y \rightarrow C$ and $\pi': C \rightarrow Z$, $\|\pi'\| \leq 1$ s.t. we get the commutative diagram

$$\begin{array}{ccccc} & & & & Z \\ & & & \nearrow \pi & \uparrow T \\ A & \xrightarrow{i_A} & C & \xrightarrow{\pi'} & Z \\ \uparrow e & & \nwarrow i_Y & & \\ X & \hookrightarrow & Y & & \end{array}$$

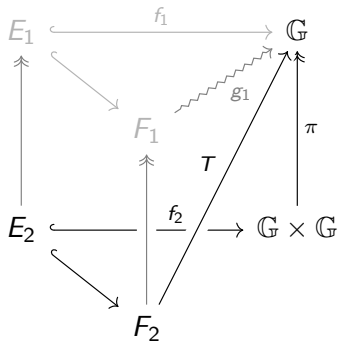


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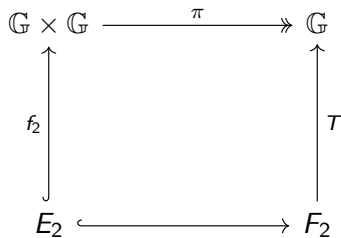
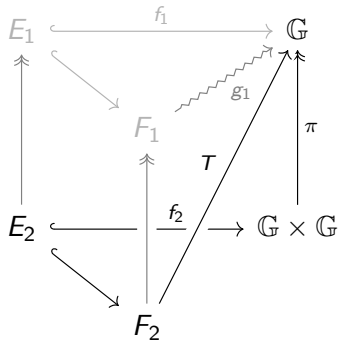


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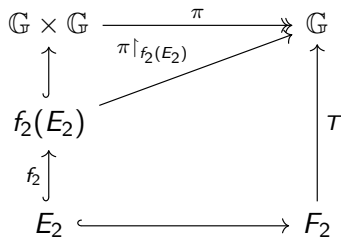
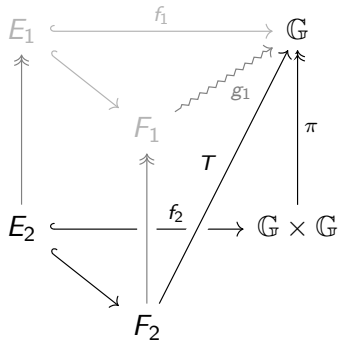


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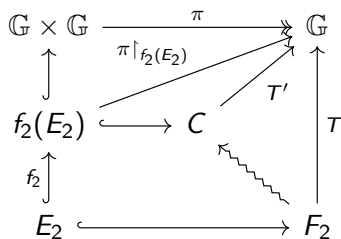
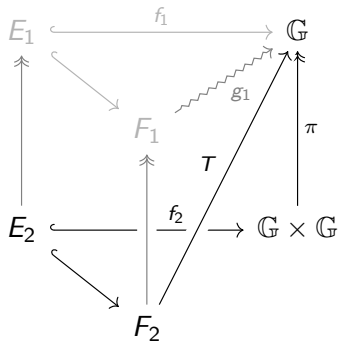


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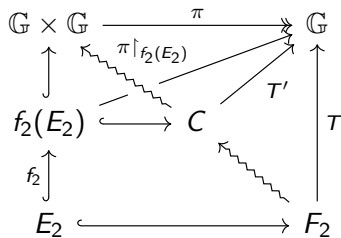
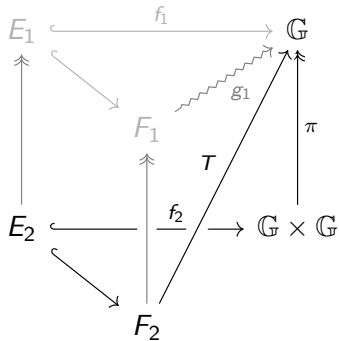
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$$\|T'\| \leq 1$$

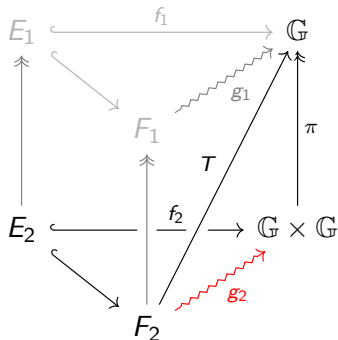


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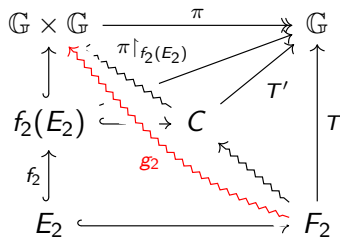


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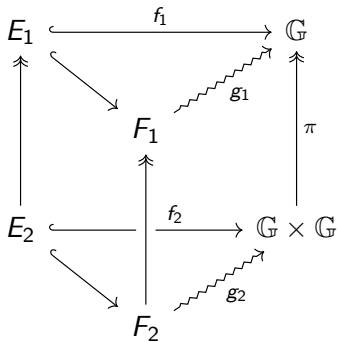
The map g_2 satisfies

$$(1 + \varepsilon_2)^{-1}(1 + \varepsilon_1)^{-1}\|x\|_2 \leq \|g_2(x)\|_2 \leq (1 + \varepsilon_2)(1 + \varepsilon_1)\|x\|_2.$$



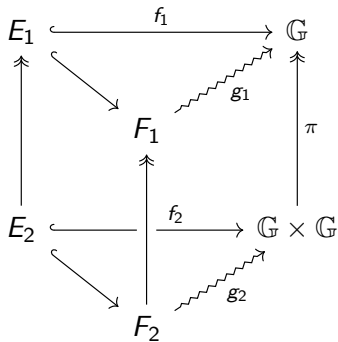


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Then we continue inductively.



- The space $\mathbb{G}^{\mathbb{N}}$ is unique up to isometries.



Properties of $\mathbb{G}^{\mathbb{N}}$ / Comparison to the Banach space case

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- It is universal for separable (graded) Fréchet spaces.





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- There is no separable Fréchet space of universal disposition (where the extension of isometric embeddings would be an isometric embedding and not just ε -isometric).



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- It is universal for separable (graded) Fréchet spaces.
- There is no separable Fréchet space of universal disposition (where the extension of isometric embeddings would be an isometric embedding and not just ε -isometric).
- No space of the form $\mathcal{C}(X)$, where X is a hemi-compact topological space, is of almost universal disposition for finite dimensional (graded) Fréchet spaces.



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