ANALOGON OF RIESZ BROTHERS THM. FOR SOBOLEV SPACES

M. WOJCIECHOWSKI
IMPAN (WARSZAWA)

$$BV = \mathring{BV}(\mathbb{R}^n) = \{ f : \nabla f \in M(\mathbb{R}^n, E) \}$$

$$\| f \|_{BV} = |\nabla f|(\mathbb{R}^n)$$

$$\mathring{W}^{1,1} \subset BV \qquad \nabla f \ll \lambda$$

PEŁCZYŃSKI, MW

$W^{1,1} \subset BV$ IS UNCOMPLEMENTED

GAGLIARDO

$$BV \ni f \longmapsto (\nabla f)_a \in L^1(\mathbb{R}^n, E)$$

IS A SURJECTION

Q: DOES THERE EXIST A RIGHT INVERSE

$$BV \ni f \longmapsto (\nabla f)_s \in M(\mathbb{R}^n; E)$$

WHAT IS THE IMAGE?

$$\dim_H (\nabla f)_s \geq n-1$$

$$\boxed{\text{THM. } \quad BV/_{W^{1,1}} \simeq M}$$

M. and F. RIESZ

$M_A = H^1$ $\qquad$ $A^* = M \oplus L^1/H^1$

PEŁCZYŃSKI

$$C^1(\mathbb{T}^n)^* \simeq M \oplus F$$

F - SEPARABLE

ALBERTI'S WORK:

$\mu \in M(\mathbb{R}^n)$ INTEGRABLE RECTIFIABLE ($\mathcal{IR}$)

IFF $\dim E(x,\mu) = 1$ μ-a.e.

$$E(x,\mu) = \left\{ \vec{v} \in E : \exists u \in BV \ \frac{|\nabla u - \vec{v} \cdot \mu|(B_{x,r})}{|\mu|(B_{x,r})} \xrightarrow[r \to \infty]{} 0 \right\}$$

<u>THM</u> (ALBERTI) $|(\nabla f)_s| \in \mathcal{IR}$

DEF. ORIENTATION $\mu \in \mathcal{F}\mathbb{R}$

$\eta = \eta_\mu : \mathbb{R}^n \to E$ S.T.

$E(x, \mu) = \operatorname{span} \eta(x)$ μ-a.e.

$|\eta(x)| = 1$

FOR $\mu = |(\nabla f)_s|$ WE CAN PUT

$$\eta_\mu = \frac{\partial (\nabla f)_s}{\partial |(\nabla f)_s|}, \quad \mu = \eta_\mu \cdot |\mu|$$

η IS WELL DEFINE IN A FOLLOWING STRONG SENSE

$\mu, \nu \in (\nabla M)_s$

IF $\mu \not\perp \nu$ i.e. $\mu|_A \overset{\ll}{\gg} \nu|_A$

THEN $\eta_\mu(x) = \pm \eta_\nu(x) \quad x \in A.$

THIS FOLLOWS FROM

$$\boxed{\begin{array}{l}\text{ALBERTI'S RANK ONE THM:} \\ \operatorname{rank}\begin{bmatrix}(\nabla u_1)(x) \\ (\nabla u_2)(x)\end{bmatrix} = 1 \qquad (|\nabla u_1| + |\nabla u_2|)\text{-a.e.}\end{array}}$$

THM $\mu \in (\nabla M)_s$, $f \in L^1(|\mu|)$

THEN $f \cdot \eta_\mu \cdot |\mu| \in (\nabla M)_s$

$(\nabla u)_s = f \cdot \eta \cdot |\mu|$, $\|u\|_{BV} \lesssim \|f\|_{L^1(\mu)}$

AN EXTENSION RESULT

PROP $B = B_{x,r}$. THERE EXISTS AN EXTENSION OPERATOR

$$\mathcal{E}_{r,x} : BV(B) \longmapsto BV(\mathbb{R}^n)$$

1) $\mathcal{E}_{r,x} f \big|_{\mathbb{R}^n \setminus B} \in W^{1,1}(\mathbb{R}^n \setminus B)$

2) $\operatorname{supp} \mathcal{E}_{r,x} f \subset B_{2r,x}$

$\| \mathcal{E}_{r,x} \|$ DOES NOT DEPEND ON x, r.

LEMMA $\phi \in M(\mathbb{R}^4)$, $\Psi \in (\nabla M)_s$

$\Phi \perp \lambda$. $\exists\, u \in BV$, $v \in (\nabla M)_s$

s.t. $\nabla u - v \perp v$; $\nabla u - v \perp \phi$

$$\| v - \Psi \| \le \frac{1}{2} \| \Psi \|$$

$$\| u \|_{BV} \le C \cdot \| \Psi \|$$

PROOF $\forall x \ \exists u_x \in BV$

$$|\nabla u_x - \psi|(B_{x,r}) \leq \frac{1}{2}|\psi|(B_{x,r})$$

FOR SUFF. SMALL r

VITALI COVERING (B_j) $B_j \cap B_{j'} = \emptyset_{\ j \neq j'}$

$$|\psi|(\mathbb{R}^n \setminus \cup B_j) = 0$$

$$u =: \sum \varepsilon_{r_j x_j} (u_{x_j} - \mathbb{E} u_{x_j})$$

$$\|u\|_{BV} \leq \sum \| \mathcal{E}_{\nu_j x_j} (u_{x_j} - \mathbb{E} u_{x_j}) \|_{BV}$$

$$\lesssim \sum \| u_{x_j} - \mathbb{E} u_{x_j} \|_{BV(B_j)}$$

$$\lesssim \sum |\nabla u_{x_j}|(B_j) \lesssim \sum |\psi|(B_j)$$

$$\|\gamma - \psi\| \leq \sum |(\nabla u_{x_j})_s - \psi|(B_j)$$

$$\leq \sum |\nabla u_{x_j} - \psi|(B_j)$$

$$\leq \frac{1}{2} \sum |\psi|(B_j) \leq \frac{1}{2} \|\psi\|$$

OTHER PROPERTIES BY CONSTRUCTION

INDUCTIVELY

a) $\psi_j = \psi_{j-1} - v_{j-1}$

b) $\psi_j \in (\nabla M)_s \quad v_j \in (\nabla M)_s$

c) $\nabla u_j - v_j \perp v_j \ , \quad \nabla u_j - v_j \perp \phi$

d) $\| v_j - \psi_j \| \leq \frac{1}{2} \| \psi_j \|$

e) $\| \nabla u_j \| \lesssim \| \psi_j \|$

WE PUT $\mathcal{X} = \sum \mathcal{X}_i$. WE GET

$\exists\, u \in BV$, $\nabla u - \psi \perp \phi$ $u = \sum u_i$

$\|\nabla u\| \leq C \cdot \|\psi\|$

WE APPLY FOR $\psi = f \cdot \eta \cdot |\mu|$ AND $\phi = \mu$.

SETTING $\Theta = \nabla u - f \cdot \eta \cdot |\mu|$

$$\boxed{\nabla u = f \cdot \eta \cdot |\mu| + \Theta \quad \text{WHEN} \quad \Theta \perp \mu \quad \text{AND} \quad \|u\|_{BV} \lesssim \|f\|_{L^1(\mu)}}$$

AN AVERAGING LEMMA:

LEMMA $B \subset \mathbb{R}^n$

$\exists\ T_B : BV(\mathbb{R}^n) \longrightarrow BV(\mathbb{R}^n)$

s.t. $T_B f|_{\mathbb{R}^n \setminus B} = f|_{\mathbb{R}^n \setminus B}$

$T_B f|_B \in W^{1,1}(B)$

$\| T_B f \|_{W^{1,1}(B)} \leqslant \| f \|_{BV(B)}$.

LET F - COMPACT SUPPORT of $|M|$

s.t. $|\Theta|(F) = 0$

$$\mathbb{R}^n \setminus F = \bigcup_{i=1}^{n+1} \cup B_n^i \qquad B_n^i \cap B_m^i = \phi \quad n \neq m$$

$$T^i f(x) = \begin{cases} T_{B_n^i} f(x) & x \in B_n^i \\ x & \text{OTHERWISE} \end{cases}$$

$$T^1 \circ T^2 \circ \dots \circ T^{n+1} \cancel{f} u = f \cdot \eta \cdot |M| + g$$

$$g \in W^{1,1}, \quad \|g\|_{W^{1,1}} \lesssim \|f\|_{L^1(|M|)}.$$

IT FOLLOWS THAT

$$\|(\nabla f)_s\|_{BV/W^{11}} \sim \|(\nabla f)_s\|_M$$

$$\left((\nabla M)_s , \|\cdot\|_M \right) \simeq BV/W^{11}$$

WE DEFINE BY INDUCTION

$$(\nabla M)_s = X_{\mu_1} \oplus_1 X_{\mu_2} \oplus \cdots \oplus X_{\mu_k} \oplus_1 Y_k$$

WHERE $\mu_1, \ldots, \mu_k \perp \nabla u$ FOR $\nabla u \in Y_k$

$$\mu_{k+1} = |\nabla u| \quad \text{FOR SOME } \nabla u \in Y_k$$

$$T_{\mu_{k+1}} : Y_k \to Y_k$$

$$T_{\mu_{k+1}} v = \frac{\partial v}{\partial |\mu_{k+1}|}$$

$$v = T_{\mu_{k+1}} v + (I - T_{\mu_{k+1}}) v$$

LEBESGUE DECOMPOSITION

$$\operatorname{Im}(T_{\mu_{k+1}}) \simeq L^1(|\mu_{k+1}|)$$

$$\operatorname{Im}(I - T_{\mu_{k+1}}) = Y_{k+1}$$

THEN

$$(\nabla M)_s \simeq \Big(\bigoplus_{\alpha\in\Gamma} X_{\mu_\alpha}\Big)_{\ell^1}$$

$$X_{\mu_\alpha} \simeq L^1[0,1)$$